Math 266 Second Exam Review Notes

Techniques of Integration

A. Integration by Parts $\int u\,dv = uv - \int v\,du$

Choose dv to be the most complicated part of the integrand that you can integrate

Tabular Integration - compact way of doing integration by parts repeatedly
- *u* eventually becomes 0 after repeated differentiation

Ex. $\int x^3 e^x dx$ $\int (x+1)^4 \sin 3x\,dx$

B. Trigonometric Integrals

1. $\int \sin^n x\,dx$ or $\int \cos^n x\,dx$

Case a. *n* is an odd positive integer

Write $\int \sin^n x\,dx = \int \sin^{n-1} x \sin x\,dx$ and use $\sin^2 x = 1 - \cos^2 x$

Write $\int \cos^n x\,dx = \int \cos^{n-1} x \cos x\,dx$ and use $\cos^2 x = 1 - \sin^2 x$

Case b. *n* is an even positive integer

Use $\sin^2 x = \dfrac{1-\cos 2x}{2}$ and $\cos^2 x = \dfrac{1+\cos 2x}{2}$

2. $\int \sin^m x \cos^n x\,dx$

Case a. *m* is an odd positive integer

Write $\int \sin^m x \cos^n x\,dx = \int \sin^{m-1} x \cos^n x \cdot \sin x\,dx$

and use $\sin^2 x = 1 - \cos^2 x$

Case b. *n* is an odd positive integer

Write $\int \sin^m x \cos^n x\,dx = \int \sin^m x \cos^{n-1} x \cdot \cos x\,dx$

and use $\cos^2 x = 1 - \sin^2 x$

Case c. *m* and *n* are both even

Use $\sin^2 x = \dfrac{1-\cos 2x}{2}$ and $\cos^2 x = \dfrac{1+\cos 2x}{2}$

3. $\int \tan^m x \sec^n x\,dx$

Case a. *m* is an odd positive integer

Write $\int \tan^m x \sec^n x\,dx = \int \tan^{m-1} x \sec^{n-1} x \cdot \sec x \tan x\,dx$

and use $\tan^2 x = \sec^2 x - 1$

Case b. *n* is an even positive integer

Write $\int \tan^m x \sec^n x\,dx = \int \tan^m x \sec^{n-2} x \cdot \sec^2 x\,dx$

and use $\sec^2 x = 1 + \tan^2 x$

Case c. *m* even and *n* odd

No standard method. Use integration by parts?

4. $\int \tan^n x\,dx$ or $\int \cot^n x\,dx$

ex. $\int \tan^7 x\,dx = \int \left(\tan^7 x + \tan^5 x - \tan^5 x + \tan^3 x - \tan^3 x + \tan x - \tan x\right) dx$

and group as follows: 1st and 2nd, 3rd and 5th, 4th and 6th and use

$\tan^2 x + 1 = \sec^2 x$

C. Trigonometric Substitutions

$$\sqrt{a^2 - x^2} \qquad x = a\sin\theta$$
$$\sqrt{a^2 + x^2} \qquad x = a\tan\theta$$
$$\sqrt{x^2 - a^2} \qquad x = a\sec\theta$$

D. Integrals of Rational Functions

$q(x) = f(x)/g(x)$, where $f(x)$ and $g(x)$ are polynomials

1. degree of $f \geq$ degree of g

- use long division first

2. degree of $f <$ degree of g

- write $g(x)$ as a product of linear factors $ax + b$ or irreducible quadratic factors $ax^2 + bx + c$

3. For each factor of the form $(ax+b)^n$, $n \geq 1$, the *partial fraction decomposition* contains a sum of n partial fractions of the form

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + ... + \frac{A_n}{(ax+b)^n}, A_k \text{ real.}$$

For each factor of the form $(ax^2 + bx + c)^n$, $n \geq 1$, the *partial fraction decomposition* contains a sum of n partial fractions of the form

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + ... + \frac{A_nx + B_n}{(ax^2 + bx + c)^n}, A_k \text{ real.}$$

E. Integrals Involving Quadratic Equations

Complete the square.

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x\right) + c = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$$

F. Miscellaneous Substitutions

1. radical expressions

$\sqrt[n]{f(x)}$ let $u = \sqrt[n]{f(x)}$ or $u = f(x)$

different indices : ex, $\sqrt{x}$, $\sqrt[3]{x}$ let $u = \sqrt[6]{x}$

2. rational expression in $\sin x$ and $\cos x$

Let $u = \tan\dfrac{x}{2}$ for $-\pi < x < \pi$. (1)

Then $\cos\dfrac{x}{2} = \dfrac{1}{\sqrt{1+u^2}}$ and $\sin\dfrac{x}{2} = \dfrac{u}{\sqrt{1+u^2}}$

thus $\sin x = \dfrac{2u}{1+u^2}$ (2)

and $\cos x = \dfrac{1-u^2}{1+u^2}$ (3)

Also, since $x = 2\tan^{-1}u$ then $dx = \dfrac{2}{1+u^2}du$ (4)