Math 266 First Exam Review Notes

A. Inverse Functions

A function $f: D \to R$ is **one-to-one** if $a \neq b$ in $D \Rightarrow f(a) \neq f(b)$ in R.
f is **increasing** $\Rightarrow f$ is one-to-one.
f is **decreasing** $\Rightarrow f$ is one-to-one.

Let $f: D \to R$ be one-to-one. A function $g: R \to D$ is the **inverse function** of f, provided that

$$\forall x \in D \text{ and } \forall y \in R: \; y = f(x) \Leftrightarrow x = g(y)$$

Procedure for finding the inverse:

1. Verify that $f: D \to R$ is one-to-one.
2. Solve $y = f(x)$ for x.
3. Switch the variables x and y. Let $y = g(x)$.
4. Show **(i)** $g(f(x)) = x, \; \forall x \in D$
(ii) $f(g(y)) = y, \; \forall y \in R$

Inverse Function Theorem If (i) f is differentiable
(ii) $g = f^{-1}$ exists
(iii) $f'(g(c)) \neq 0$
then g is differentiable at c and $g'(c) = \dfrac{1}{f'(g(c))}$.

B. The ln and e Functions

The **natural logarithm function**, denoted by **ln**, is defined by

$$\ln x = \int_1^x \frac{1}{t}\,dt \qquad \text{for every } x > 0.$$

Note that $-\infty < \ln x < \infty$, or the range is the set of real numbers.

Derivative

(i) $D_x \ln u = \dfrac{1}{u} D_x u$ if $g(x) > 0$ (ii) $D_x \ln|u| = \dfrac{1}{u} D_x u$ if $g(x) \neq 0$

Laws of Natural Logarithms If $p > 0$ and $q > 0$, then

(i) $\ln pq = \ln p + \ln q$
(ii) $\ln \dfrac{p}{q} = \ln p - \ln q$
(iii) $\ln p^r = r \ln p$ for every rational number r

Logarithmic Differentiation

- useful when the function is complicated, or
- for functions where both the base and exponent are variable (ex, x^x)

Steps:

1. $y = f(x)$ (given)
2. $\ln y = \ln f(x)$ (take natural log and simplify)

3. $D_x[\ln y] = D_x[\ln f(x)]$ (differentiate implicitly)

4. $\frac{1}{y} D_x y = D_x[\ln f(x)]$

5. $D_x y = f(x) D_x[\ln f(x)]$ (multiply by $y = f(x)$)

The **natural exponential function** is the inverse of the natural logarithmic function.

$$y = \exp x \Leftrightarrow x = \ln y,\ x \in \mathbb{R},\ y > 0$$

Properties: (i) $e^p e^q = e^{p+q}$ (ii) $\frac{e^p}{e^q} = e^{p-q}$ (iii) $(e^p)^r = e^{pr}$

Derivative If $u = g(x)$ and g is differentiable, then $D_x e^u = e^u D_x u$.

Integration Formulas for e and ln:

$$\int \frac{1}{u} du = \ln|u| + C \quad \text{and} \quad \int e^u\, du = e^u + C$$

Integration Formulas for tan, cot, sec, csc:

(i) $\int \tan u\, du = -\ln|\cos u| + C$

(ii) $\int \cot u\, du = \ln|\sin u| + C$

(iii) $\int \sec u\, du = \ln|\sec u + \tan u| + C$

(iv) $\int \csc u\, du = \ln|\csc u - \cot u| + C$

C. General Exponential and Log Functions

If a is a positive real number and x is any real number, then

$$a^x = e^{x \ln a}$$

If $f(x) = a^x$, then f is the **exponential function with base a**.

Properties $a^u a^v = a^{u+v}$, $(a^u)^v = a^{uv}$, $(ab)^u = a^u b^u$

$$\frac{a^u}{a^v} = a^{u-v}, \qquad \left(\frac{a}{b}\right)^u = \frac{a^u}{b^u}$$

Derivative $D_x a^u = (a^u \ln a) D_x u$

Integral $\int a^u\, du = \left(\frac{1}{\ln a}\right) a^u + C$

If $a \neq 1$ and $f(x) = a^x$, then f is a one-to-one function. Its inverse function $\log_a$ is the **logarithmic function with base a**. We have

$$y = \log_a x \Leftrightarrow x = a^y$$

The expression $\log_a x$ is called the **logarithm of x with base a**.

Relationship Between $\log_a$ **and** $\ln$**:** $\log_a x = \frac{\ln x}{\ln a}$

Derivative $D_x \log_a |u| = D_x \left(\frac{\ln|u|}{\ln a} \right) = \frac{1}{\ln a} \cdot \frac{1}{u} D_x u$

Integral same as ln formula for $\int \frac{du}{u}$

D. Laws of Growth and Decay

Let y be a differentiable function of t such that $y > 0$ for every t, and let y_0 be the value of y at $t = 0$. If

$$dy/dt = cy$$

for some constant c, then

$$y = y_0 e^{ct}. \quad (*)$$

If $c > 0$ then $(*)$ is a **law of growth.** (ex, bacterial population)

If $c < 0$, then $(*)$ is a **law of decay**. (ex, radioactive decay)

Newton's Law of Cooling

$dT/dt = c(T - S)$, where T is the temperature of the object at any time t and S is the temperature of the surrounding medium (ex, air)

E. Inverse Trigonometric Functions

1. $y = \sin^{-1} x \Leftrightarrow \sin y = x$, where $-1 \le x \le 1$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
2. $y = \cos^{-1} x \Leftrightarrow \cos y = x$, where $-1 \le x \le 1$ and $0 \le y \le \pi$
3. $y = \tan^{-1} x \Leftrightarrow \tan y = x$, where $-\infty \le x \le \infty$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$
4. $y = \csc^{-1} x \Leftrightarrow \csc y = x$, where $|x| \ge 1$ and $y \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$
5. $y = \sec^{-1} x \Leftrightarrow \sec y = x$, where $|x| \ge 1$ and $y \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$
6. $y = \cot^{-1} x \Leftrightarrow \cot y = x$, where $-\infty < x < \infty$ and $0 < y < \pi$

Derivatives

$$D_x \sin^{-1} u = \frac{D_x u}{\sqrt{1 - u^2}}$$

$$D_x \cos^{-1} u = -\frac{D_x u}{\sqrt{1 - u^2}}$$

$$D_x \tan^{-1} u = \frac{D_x u}{1 + u^2}$$

$$D_x \sec^{-1} u = \frac{D_x u}{u\sqrt{u^2 - 1}}$$

Integrals

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1} \frac{u}{a} + C$$

$$\int \frac{du}{a^2 + u^2} = \frac{1}{a} \tan^{-1} \frac{u}{a} + C$$

$$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \sec^{-1} \frac{u}{a} + C$$

F. Hyperbolic Functions

1. $\sinh x = \dfrac{e^{x} - e^{-x}}{2}$

2. $\cosh x = \dfrac{e^{x} + e^{-x}}{2}$

3. $\tanh x = \dfrac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$

4. $\operatorname{csch} x = \dfrac{2}{e^{x} - e^{-x}}$

5. $\operatorname{sech} x = \dfrac{2}{e^{x} + e^{-x}}$

6. $\coth x = \dfrac{e^{x} + e^{-x}}{e^{x} - e^{-x}}$

"Pythagorean Identities"

$$\cosh^2 x - \sinh^2 x = 1, \quad 1 - \tanh^2 x = \operatorname{sech}^2 x, \quad \coth^2 x - 1 = \operatorname{csch}^2 x$$

Derivatives

$D_x \sinh u = \cosh u \, D_x u$

$D_x \cosh u = \sinh u \, D_x u$

$D_x \tanh u = \operatorname{sech}^2 u \, D_x u$

$D_x \coth u = -\operatorname{csch}^2 u \, D_x u$

$D_x \operatorname{sech} u = -\operatorname{sech} u \tanh u \, D_x u$

$D_x \operatorname{csch} u = -\operatorname{csch} u \coth u \, D_x u$

Integrals

$\int \cosh u \, du = \sinh u + C$

$\int \sinh u \, du = \cosh u + C$

$\int \operatorname{sech}^2 u \, du = \tanh u + C$

$\int \operatorname{csch}^2 u \, du = -\coth u + C$

$\int \operatorname{sech} u \tanh u \, du = -\operatorname{sech} u + C$

$\int \operatorname{csch} u \coth u \, du = -\operatorname{csch} u + C$