Math 270 Exam 3 Review Answer Key Draft 2

1. T F T F F

2. a. Area of $\Delta ABC = \dfrac{\sqrt{30}}{2}$ b. Area of $\square = \sqrt{30}$
   c. Equation of plane: $x + 2y - 7 = 0$
   d. Parametric equations of line: $x = t; \quad y = 3 + 2t; \quad z = 0$

3. a. $\theta = 8.2^\circ$ b. $a = -\dfrac{5}{3}$ c. $\cos 1$ d. $\left\{ \sqrt{3}\,t, \dfrac{2\pi \sin 2\pi t + 3t}{\sqrt{2\pi^2 - 3}} \right\}$

4. Let $S = \left\{ \vec{v_1}, \vec{v_2}, \vec{v_3} \right\}$ be nonzero vectors s.t. any two vectors are orthogonal. Suppose $a_1\vec{v_1} + a_2\vec{v_2} + a_3\vec{v_3} = \vec{0}$. Then
$0 = \left(\vec{v_1}, \vec{0}\right) = \left(\vec{v_1}, a_1\vec{v_1} + a_2\vec{v_2} + a_3\vec{v_3}\right) = a_1\left(\vec{v_1}, \vec{v_1}\right) + a_2\left(\vec{v_1}, \vec{v_2}\right) + a_3\left(\vec{v_1}, \vec{v_3}\right) = a_1\left(\vec{v_1}, \vec{v_1}\right)$
since $\left(\vec{v_1}, \vec{v_2}\right) = 0$ and $\left(\vec{v_1}, \vec{v_3}\right) = 0$ because of orthogonality. Since $\left(\vec{v_1}, \vec{v_1}\right) > 0$, $a_1 = 0$. In a similar way, we can show that $a_2 = a_3 = 0$. Thus S is linearly independent.

5. $\begin{bmatrix} 15 \\ 3 \\ 3 \end{bmatrix}_S = \begin{bmatrix} 9 \\ 9 \\ 9 \end{bmatrix}$

6. a. $\left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \end{bmatrix} \right\}$

   b. $\left\{ \begin{bmatrix} -\frac{5}{\sqrt{30}} \\ \frac{2}{\sqrt{30}} \\ \frac{1}{\sqrt{30}} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{2}{\sqrt{30}} \\ \frac{5}{\sqrt{30}} \\ 0 \\ \frac{1}{\sqrt{30}} \end{bmatrix} \right\}$ c. $\left\{ \begin{bmatrix} \frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{35}} \\ -\frac{3}{\sqrt{35}} \\ \frac{5}{\sqrt{35}} \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{210}} \\ \frac{3}{\sqrt{210}} \\ \frac{2}{\sqrt{210}} \\ \frac{14}{\sqrt{210}} \end{bmatrix} \right\}$

   d. $\left\{ 1, \sqrt{3}\left(2t - 1\right), \sqrt{5}\left(6t^2 - 6t + 1\right) \right\}$

7. a. Yes b. No c. No d. Yes e. Yes f. Yes g. No h. Yes

8. a. $\begin{bmatrix} 1 & -2 \\ 3 & -1 \\ 0 & 4 \end{bmatrix}$ b. $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \\ 1 & -1 \end{bmatrix}$

9. Let X and Y be in M_{33} and c be any real scalar.
$$L(X+Y) = A(X+Y) - (X+Y)A = AX + AY - XA - YA = (AX - XA) + (AY - YA) = L(X) + L(Y)$$
$$L(cX) = A(cX) - (cX)A = c(AX) - c(XA) = c(AX - XA) = cL(X)$$
Thus, L is a linear transformation.

10. Let f and g be differentiable functions. Then
$L(f+g) = (f+g)' = f' + g' = L(f) + L(g)$. Also, for any scalar c,
$L(cf) = (cf)' = cf'$. Therefore, L is a linear transformation.

11. a. $\begin{bmatrix} 3 \\ -1 \\ 7 \end{bmatrix}$ b. $\begin{bmatrix} -2 \\ 0 \\ -18 \end{bmatrix}$

12. Let f and g be integrable over $[a,b]$. Then
$L(f+g) = \int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx = L(f) + L(g)$. Also for scalar c $L(cf) = \int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx = cL(f)$. Thus L is a linear transformation.

13. a. Code: 19 20 21 4 25 8 1 18 4 b. ?

14. a. Basis for Ker L = $\{t^2 + t + 1\}$; dim Ker L =1
Basis for range L = $\{t^2, t\}$; dim range L = 2
b. Basis for Ker L = $\{2t - 1\}$; dim Ker L =1
Basis for range L = $\{1\}$; dim range L =1
c. Basis for Ker L = $\left\{\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}\right\}$; dim Ker L =1
Basis for range L = $\left\{\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}\right\}$; dim range L =3

15. Proof: Suppose that A and B are nonsingular matrices. Then A^{-1} and B^{-1} exists and $AB = B^{-1}(BA)B$ and $BA = A^{-1}(AB)A$. Therefore AB and BA are similar.

16. Proof: Suppose A and B are similar matrices. Then, there exists a nonsingular matrix P such that $B = P^{-1}AP$. We have

$$B^2 = \left(P^{-1}AP\right)^2 = \left(P^{-1}AP\right)\left(P^{-1}AP\right) = \left(P^{-1}A\left(PP^{-1}\right)AP\right) = P^{-1}AI_nAP = P^{-1}A^2P.$$

Now suppose that for any positive integer $k, B^k = P^{-1}A^kP,$ then ,

$$B^{k+1} = B^k\left(P^{-1}AP\right) = \left(P^{-1}A^kP\right)\left(P^{-1}AP\right) = \left(P^{-1}A^k\left(PP^{-1}\right)AP\right) = P^{-1}A^kI_nAP = P^{-1}A^{k+1}P$$

By induction $B^k = P^{-1}A^kP$ for any positive integer k.

17. Proof: Suppose $\left\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\right\}$ spans V. Let $\vec{w}$ be in the range of L, that is, $L\left(\vec{v}\right) = \vec{w}$ for some $\vec{v}$ in V. But $\vec{v} = a_1\vec{v_1} + a_2\vec{v_2} + \ldots + a_k\vec{v_k}$ for some scalars $a_1, a_2, \ldots a_k$. Thus, we have

$$\vec{w} = L\left(\vec{v}\right) = L\left(a_1\vec{v_1} + a_2\vec{v_2} + \ldots + a_k\vec{v_k}\right) = a_1L\left(\vec{v_1}\right) + a_2L\left(\vec{v_2}\right) + \ldots + a_kL\left(\vec{v_k}\right).$$

This shows that $\left\{L\left(\vec{v_1}\right), L\left(\vec{v_2}\right), \ldots, L\left(\vec{v_k}\right)\right\}$ spans the range of L.

18. Proof: $L: V \to W$; $L(f) = f'$. If f is in the Ker L, then $f'(x) = 0$ which implies that $f(x) = k$, a constant function. So Ker L = $\left\{f : f(x) = k \;\; \forall x\right\}$. L is not one-to-one because Ker L $\neq \left\{\vec{0}\right\}$. L is not onto since the function e^{x^2} is not in the range of L, i.e. there is no function f for which $f'(x) = e^{x^2}$.

19. a. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$ b. $\begin{bmatrix} 1 & 1 \\ \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{3}{2} \end{bmatrix}$

20. $\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$

21. $\begin{bmatrix} -3 & 4 \\ 2 & -3 \end{bmatrix}$

22. a. $L\left(p_1(t)\right) = 3t - 3$; $L\left(p_2(t)\right) = -t + 8$

b. $\left[L\left(p_1(t)\right)\right]_S = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$; $\left[L\left(p_2(t)\right)\right]_S = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ c. $L(t+2) = -\frac{7}{3}t + \frac{35}{3}$