Math 270 **Exam 3 Review** **Draft 3**

1. Mark the following as TRUE or FALSE.
   a. Let $L: V \to W$ be a linear transformation. The L is one-to-one if and only if dim range L = dim V.
   b. An orthonormal set can contain the zero vector.
   c. If $\vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{w}$, then $\vec{v}$ is orthogonal to every vector in the span $\{\vec{u}, \vec{w}\}$.
   d. If $L: R^3 \to R^5$ is a linear transformation, then it is possible for L to be an onto transformation.
   e. The determinant of 2×2 matrices defines a linear transformation from M_{22} to R^1.

2. Let A (3, 2, -1), B (5, 1, 0), C (0, 3, 0), D (1, 3, 2). Find the following using cross products of vectors.
   a. Area of ΔABC
   b. Area of parallelogram with adjacent sides $\overrightarrow{AB}$ and $\overrightarrow{AC}$
   c. Equation of the plane containing points B, A, and D.
   d. Parametric equations of the line through C perpendicular to the plane through B, A, and D.

3. Use the inner product space of continuous functions on [0,1] defined as $(f, g) = \int_0^1 f(t)g(t)dt$ for the following.
   a. Find the angle (in degrees) between $p(t) = 1$ and $q(t) = 2t + 3$.
   b. Let $p(t) = 6t + 2$ and $q(t) = at + 1$. For what values of a are $p(t)$ and $q(t)$ are orthogonal?
   c. Find the distance between $\sin t$ and $\cos t$.
   d. Let $S = \{t, \sin 2\pi t\}$ be a basis for a subspace W of the inner product space. Find an orthonormal basis for W.

4. Let $S = \{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ be a set of nonzero vectors in R^3 such that any two vectors are orthogonal. Prove that S is linearly independent.

5. Verify that $S = \left\{ \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}, \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{bmatrix}, \begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{bmatrix} \right\}$ is an orthonormal basis for R^3. Find the coordinate vector of $\vec{v} = \begin{bmatrix} 15 \\ 3 \\ 3 \end{bmatrix}$ with respect to S.

6. Use the Gram-Schmidt process to find an orthonormal basis for the following.
   a. Subspace of R_4 with basis $\{[1 \quad 0 \quad -1 \quad 0], [1 \quad -1 \quad 0 \quad 0], [2 \quad 1 \quad 0 \quad 0]\}$
   b. Null space of $A = \begin{bmatrix} 1 & 0 & 5 & -2 \\ 0 & 1 & -2 & -5 \end{bmatrix}$
   c. Subspace of R^4 consisting of all vectors of the form $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ such that $a - 3b - 2c + d = 0$
   d. Euclidean space P_2 with basis $\{1, t, t^2\}$

7. Which of the following are linear trasformations?
   a. $L: M_{nn} \to M_{nn}$ defined by $L(A) = A^T$
   b. $L: M_{nn} \to M_{nn}$ defined by $L(A) = A^{-1}$
   c. $L: M_{nn} \to R^1$ defined by $L(A) = \det(A)$
   d. $L: M_{nn} \to R^1$ defined by $L(A) = Tr(A)$
   e. $L: R^2 \to R^2$ defined by $L\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = \begin{bmatrix} u_1 + ku_2 \\ u_2 \end{bmatrix}$
   f. $L: P_2 \to P_3$ defined by $L(p(t)) = t^2 p'(t) + t\, p(0)$
   g. $L: P_1 \to P_2$ defined by $L(p(t)) = t\, p(t) + 1$
   h. $L: C[0,1] \to R^1$ defined by $L(f) = \int_0^1 f(t)dt$

8. Find the standard matrix representing each given linear transformation.
   a. $L: R^2 \to R^3$ defined by $L\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = \begin{bmatrix} u_1 - 2u_2 \\ 3u_1 - u_2 \\ 4u_2 \end{bmatrix}$
   b. $L: P_1 \to P_2$ defined by $L(at + b) = \frac{1}{2}at^2 + bt + (a - b)$

9. Let A be a fixed 3×3 matrix, $L: M_{33} \to M_{33}$ be defined by $L(X) = AX - XA$ for X in M_{33}. Show that L is a linear transformation.

10. Let W be the vector space of all real-valued functions and let V be the subspace of all differentiable functions. Define $L: V \to W$ by $L(f) = f'$. Prove that L is a linear transformation.

11. Let $A=\begin{bmatrix}1 & -2 & -1\\ 0 & 1 & 0\\ 3 & 4 & 1\end{bmatrix}$ be the standard matrix representing the linear transformation $L:R^3\to R^3$. Find:

a. $L\left(\begin{bmatrix}3\\ -1\\ 2\end{bmatrix}\right)$

b. $L\left(\begin{bmatrix}-5\\ 0\\ -3\end{bmatrix}\right)$

12. Let $V=C[a,b]$ be the vector space of all real-valued functions that are integrable over the interval $[a,b]$. Let W be R^1. Define $L:V\to W$ by $L(f)=\int_a^b f(x)dx$. Prove the L is a linear transformation.

13. Use the substitution scheme $A\leftrightarrow 1,\quad B\leftrightarrow 2,\quad \ldots\quad Y\leftrightarrow 25,\quad Z\leftrightarrow 26$ and encoding matrix $A=\begin{bmatrix}1 & 2 & 2\\ 2 & 3 & 3\\ 1 & 2 & 1\end{bmatrix}$.

a. Code the message STUDY HARD
b. Decode the message:
79 130 60 88 136 73 47 81 35 68 108 60 19 31 14 59 98 59

14. For each linear transformation, find a basis and the dimension for the ker L and a basis and dimension for the range L.

a. $L:P_2\to P_2$ defined by $L\left(at^2+bt+c\right)=(a+c)t^2+(b+c)t$

b. $L:P_1\to R$ defined by $L(p(t))=\int_0^1 p(t)dt$

c. $L:M_{22}\to M_{22}$ defined by $L\left(\begin{bmatrix}a & b\\ c & d\end{bmatrix}\right)=\begin{bmatrix}a+d & b-c\\ a+b & b-d\end{bmatrix}$

d. $L:R^4\to R^3$ defined by $L\left(\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}\right)=\begin{bmatrix}1 & 2 & 1 & 3\\ 2 & 1 & -1 & 2\\ 1 & 0 & 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}$

15. Prove that if A and B are nonsingular $n\times n$ matrices, then AB and BA are similar.

16. Show that if A and B are similar matrices, then A^k and B^k are similar for any positive integer k.

17. Let $L: V \to W$ be a linear transformation. If $\left\{\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots, \overrightarrow{v_k}\right\}$ spans V, show that $\left\{L\left(\overrightarrow{v_1}\right), L\left(\overrightarrow{v_2}\right), \ldots, \mathrm{L}\left(\overrightarrow{v_k}\right)\right\}$ spans range L.

18. Let W be the vector space of real-valued functions and V the subspace of all differentiable functions. Let $L: V \to W$ be the linear transformation defined by $L(f) = f'$. Find ker L. Prove or disprove that L is one-to-one. Prove or disprove that L is onto.

19. Let $L: P_1 \to P_2$ be defined by $L(p(t)) = t\,p(t) + p(0)$. Consider the ordered bases $S = \{t, 1\}$ and $S' = \{t+1, t-1\}$ for P_1, and $T = \{t^2, t, 1\}$ and $T' = \{t^2+1, t-1, t+1\}$ for P_2. Find the representation of L with respect to
    a. S and T
    b. S' and T'

20. Let V be the vector space of real-valued continuous functions with basis $S = \{e^{2t}, e^{-2t}\}$. Find the representation of the linear operator $L: V \to V$ defined by $L(f) = f'$ with respect to S.

21. Let $L: R^2 \to R^2$ be the linear transformation defined by $L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_2 - x_1 \end{bmatrix}$.

    Let S be the natural basis for R^2 and $T = \left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$. Find the representation of L with respect to S and T.

22. Let $L: P_1 \to P_1$ be a linear transformation that is represented by the matrix $A = \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}$ with respect to the basis $S = \{p_1(t), p_2(t)\}$ where $p_1(t) = t - 2$ and $p_2(t) = t + 1$. Compute
    a. $L(p_1(t))$ and $L(p_2(t))$
    b. $\left[L(p_1(t))\right]_S$ and $\left[L(p_2(t))\right]_S$
    c. $L(t+2)$