Math 265 Fifth Exam Review Notes

Applications of the Definite Integral

Area

f and *g* are continuous and $f(x) \geq g(x)$ for every *x* in $[a,b]$; to find the area *A* of the region bounded by the graphs of $f, g, x = a$, and $x = b$,

find dA = area of representative rectangle
= length $\times$ width
= $[f(x) - g(x)]\, dx$ (curve above – curve below)dx

$$A = \int_a^b dA$$

To find the area of a region bounded by $x = f(y)$, $x = g(y)$, $y = c$, and $y = d$ with $f(y) \geq g(y)$,

find dA = length $\times$ width
= $[f(y) - g(y)]dy$ (right curve – left curve)dy

$$A = \int_c^d dA$$

Note Sometimes it may be necessary to split the region.

Solids of Revolution Involving One Curve

A. Disk Method – uses circular disks **perpendicular** to the **axis of revolution**

$y = f(x)$ continuous on $[a,b]$. Revolve the region bounded by *f*, the *x*-axis, and the lines $x = a$ and $x = b$ about the *x* – axis

dV = volume of a representative circular disk
= π(radius)^2(thickness)
= $\pi[f(x)]^2\, dx$

$$V = \int_a^b dV$$

$x = f(y)$ continuous on $[c,d]$. Revolve the region bounded by *f*, the *y*-axis, and the lines $y = c$ and $y = d$ about the *y* – axis

dV = volume of a representative circular disk
= π(radius)^2(thickness)
= $\pi[f(y)]^2\, dy$

$$V = \int_c^d dV$$

B. Shell Method – uses cylindrical shells **parallel** to the axis of revolution

$y = f(x) \geq 0$ continuous on $[a,b]$. Revolve the region bounded by *f* and the lines $x = a$ and $x = b$ about the *y* – axis

dV = volume of a representative cylindrical shell
= 2π(average radius)(height)(thickness)
= $2\pi x f(x) dx$

$$V = \int_a^b dV$$

$x = f(y) \geq 0$ continuous on $[c,d]$. Revolve the region bounded by *f* and the lines $y = c$ and $y = d$ about the *x* – axis.

dV = volume of a representative cylindrical shell
= 2π(average radius)(height)(thickness)
= $2\pi y f(y) dy$

$$V = \int_c^d dV$$

Note *x* and *y* in the formulas represent the **distance from the axis of revolution**, in these cases, the *y* – axis and the *x* – axis, respectively. These need to be adjusted if the axis of revolution is a different line.

Ex.	if axis is	distance becomes
	$x = 2$	$x - 2$
	$y = -1$	$y + 1$

Solids of Revolution Involving Two Curves

A. Washer Method – disk method involving two circular disks **perpendicular** to the axis of revolution
– subtract the volume of inner disk from the volume of outer disk

$y = f(x) \geq 0$ and $y = g(x) \geq 0$ continuous on $[a,b]$ with $f \geq g$. Revolve the region bounded by *f*, *g*, and the lines $x = a$ and $x = b$ about the *x* – axis

dV = volume of a representative washer
= volume of outer disk – volume of inner disk
= π(outer radius)^2(thickness) $-\pi$(inner radius)^2(thickness)
= $\pi[f(x)]^2 dx - \pi[g(x)]^2 dx$
= $\pi\left\{[f(x)]^2 - [g(x)]^2\right\} dx$

$$V = \int_a^b dV$$

B. Shell Method – same formula as in the case for one curve, except that the height becomes

$f(x) - g(x)$ above – below

or $f(y) - g(y)$ right – left

Volumes by Cross Sections

Let S be a solid bounded by planes that are perpendicular to the x – axis at a and b. If, for every x in $[a,b]$, the cross-sectional area of S is given by $A(x)$, where A is continuous on $[a,b]$, then the volume of S is

$$V = \int_a^b A(x)dx$$

Arc Length

Let $y = f(x)$ be smooth on $[a,b]$. To find the arc length of the graph of f from $A(a,f(a))$ to $B(b,f(b))$.

$ds =$ length of a "little" arc; the differential of arc length s

$$ds = \sqrt{(dx)^2 + (dy)^2}$$

$$= \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

$$ds = \sqrt{1 + [f'(x)]^2}\ dx$$

$$s = \int_a^b ds$$

Let $x = g(y)$ be smooth on $[c,d]$. To find the arc length of the graph of f from $(g(c),c)$ to $(g(d),d)$.

$$ds = \sqrt{(dx)^2 + (dy)^2}$$

$$= \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\ dy$$

$$ds = \sqrt{1 + [g'(y)]^2}\ dy$$

$$s = \int_c^d ds$$

Surfaces of Revolutions

$y = f(x) \geq 0$ is smooth on $[a,b]$. Find the area S of the surface generated by revolving the graph of f about the x – axis.

$dS =$ area of surface when a small arc is revolved about the x – axis

$= 2\pi$(average radius)(slant height)   *surface area of frustum of a cone*

$= 2\pi f(x)ds$

$$dS = 2\pi f(x)\sqrt{1 + [f'(x)]^2}\ dx$$

$$S = \int_a^b dS$$

$x = g(y) \geq 0$ is smooth on $[c, d]$. Find the area *S* of the surface generated by revolving the graph of *f* about the *y* – axis.

$$dS = \text{ area of surface when a small arc is revolved about the } y - \text{axis}$$
$$= 2\pi(\text{average radius})(\text{slant height}) \qquad \textit{surface area of frustum of a cone}$$
$$= 2\pi g(y) ds$$
$$dS = 2\pi g(y)\sqrt{1 + [g'(y)]^2}\ dy$$
$$S = \int_c^d dS$$

<u>Work</u>

<u>Definition</u> If a constant force *F* acts on an object, moving at a distance *d* in the direction of the force, the **work** *W* done is

$$W = Fd.$$

Determine the work done by a **variable force** in moving an object rectilinearly in the same direction as the force.

If $f(x)$ is the force at *x* and if *f* is continuous on $[a, b]$, find the work *W* done in moving an object along the *x* – axis from $x = a$ to $x = b$.

$$dW = (\text{force})(\text{distance})$$
$$= f(x)dx \qquad \text{work done in moving an object a distance } dx$$
$$W = \int_a^b dW$$

Similar result holds for moving an object along the *y* – axis.

<u>Moments and Centers of Mass</u>

moment = mass $\times$ distance

point-mass – assume the mass of an object is concentrated at a point

A. Let *S* be a system of point-masses $m_1, m_2, \ldots, m_n$ located at $x_1, x_2, \ldots, x_n$ on a coordinate line, and let $m = \sum_{k=1}^{n} m_k$ denote the total mass.

(i) The **moment of *S* about the origin** is $M_0 = \sum_{k=1}^{n} m_k x_k$.

(ii) The **center of mass** (or **center of gravity**) of *S* is $\bar{x} = \dfrac{M_0}{m}$.

B. Let S denote a system of point-masses $m_1, m_2, \ldots, m_n$ located at $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ in a coordinate plane, and let $m = \sum_{k=1}^{n} m_k$ denote the total mass.

(i) The **moment of *S* about the *x* – axis** is $M_x = \sum_{k=1}^{n} m_k y_k$.

(ii) The **moment of *S* about the *y* – axis** is $M_y = \sum_{k=1}^{n} m_k x_k$.

(iii) The **center of mass** of S is the point $(\bar{x}, \bar{y})$ such that

$$\bar{x} = \frac{M_y}{m}, \quad \bar{y} = \frac{M_x}{m}.$$

C. Let L be a lamina with area density ρ and in the shape of a region bounded by *f, g, x = a, x = b*

(i) The **mass** of L is $m = \int_a^b dm$,

where

$$dm = \text{(density)(area)}$$
$$= \rho[f(x) - g(x)]dx$$

(ii) The **moments of *L* about the *x* – axis and *y* – axis** are

$$M_x = \int_a^b dM_x \quad \text{and} \quad M_y = \int_a^b dM_y ,$$

where

$$dM_x = \text{(mass)(distance from the } x \text{ – axis)}$$
$$= \rho[f(x) - g(x)]dx \cdot \tfrac{1}{2}[f(x) + g(x)]$$
$$= \tfrac{\rho}{2}[f(x) + g(x)][f(x) - g(x)]dx$$

$$dM_y = \text{(mass)(distance from the } y \text{ – axis)}$$
$$= \rho[f(x) - g(x)]dx \cdot x$$
$$= \rho x[f(x) - g(x)]dx$$

(iii) The **center of mass** of L is the point $(\bar{x}, \bar{y})$ such that

$$\bar{x} = \frac{M_y}{m}, \quad \bar{y} = \frac{M_x}{m}.$$

<u>Definition</u> The center of mass of a plane region is called the **centroid** of the region.

<u>Theorem of Pappus</u>

Let R be a region in a plane that lies entirely on one side of a line l in the plane. If R is revolved once around l, the volume of the resulting solid is the product of the area of R and the distance traveled by the centroid of R.

Applications of the Definite Integral

1. **Area of a Region**
   Formula: length x width, where
   length = (top y – bottom y) or (right x – left x)
   width = dx or dy

2. **Volume of a Solid of Revolution**
   a. Disk – perpendicular to axis of revolution; no hole
   Formula: $\pi r^2 h$, where **radius** r = (top y – bottom y) or (right x – left x)
   and **thickness** h = dx or dy

   b. Washer – perpendicular to axis of revolution; with hole
   Formula: $\pi R^2 h - \pi r^2 h$, where R, r, and h are defined as in 2a

   c. Shell – parallel to axis of revolution
   Formula: $2\pi \bar{r}\, h t$, where

   average radius $\bar{r} = \begin{cases} x \text{ or } y & \text{(distance from the } y-axis \text{ or } x-axis) \\ x-a,\ a-x,\ y-b, \text{ or } b-y & \text{(distance from the line } x=a \text{ or } y=b) \end{cases}$

   height h = (top y – bottom y) or (right x – left x)
   thickness t = dx or dy

   d. Pappus' Theorem
   Formula: Ad, where
   area A can be obtained as in #1
   distance d = distance traveled by the centroid $(\bar{x}, \bar{y})$ (see # 7 below)

3. **Volume by Cross Sections**
   Formula: Ah, where A is the **area of a cross section**
   and h = dx or dy

4. **Arc Length**
   Formula: $ds = \sqrt{(dx)^2 + (dy)^2}$

   which becomes $ds = \sqrt{1 + [f'(x)]^2}\, dx$

   or $ds = \sqrt{1 + [g'(y)]^2}\, dy$

5. **Area of a Surface of Revolution**
   Formula: $dS = 2\pi \bar{r}\, ds$, where

   average radius $\bar{r} = \begin{cases} x \text{ or } y & \text{(distance from the } y-axis \text{ or } x-axis) \\ x-a,\ a-x,\ y-b, \text{ or } b-y & \text{(distance from the line } x=a \text{ or } y=b) \end{cases}$

   and ds is as defined in #4

6. **Work**
   Formula: force x distance

   a. Spring Compression or Stretching
   Formula: force x distance, where force = kx and distance = dx

   b. Lifting a Cable, Rope, Bucket, etc.
   Formula: force x distance, where force = $F(y)$ (need to figure this out) and distance = dy

c. Pumping Liquids out of a Tank

Formula: force x distance, where force = $F(y)dy$ (has something to do with the shape of the tank, ex. cylinder)

and distance = $h - y$ (h is the height the liquid is pumped out)

7. Center of Mass or Centroid $(\overline{x}, \overline{y})$

Formulas: mass = density x area

moment = mass x distance

$$\overline{x} = \frac{M_y}{m} = \frac{\text{moment about the } y-\text{axis}}{\text{total mass}}$$

$$\overline{y} = \frac{M_x}{m} = \frac{\text{moment about the } x-\text{axis}}{\text{total mass}}$$