Math 265 Fourth Exam Review Notes

Integrals

Definition A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for every x in I. The process of finding F is called **antidifferentiation**.

Theorem Let F be an antiderivative of f on an interval I. If G is any antiderivative of f on I, then

$$G(x) = F(x) + C$$

for some constant c and every x in I.

Definition The notation

$$\int f(x)dx = F(x) + C$$

where $F'(x) = f(x)$ and C is an arbitrary constant, denotes the family of all antiderivatives of $f(x)$ on an interval I.

Theorem **(i)** $\int [D_x f(x)]dx = f(x) + C$

(ii) $D_x \left[\int f(x)dx \right] = f(x)$

Theorem **(i)** $\int cf(x)dx = c\int f(x)dx$ for any constant c

(ii) $\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$

Method of Substitution

If F is an antiderivative of f, then

$$\int f(g(x))g'(x)dx = F(g(x)) + C .$$

If $u = g(x)$ and $du = g'(x)dx$, then

$$\int f(u)du = F(u) + C .$$

Summation Notation $\sum_{k=1}^{n} a_k = a_1 + a_2 + \ldots + a_n$

The letter k is the **index of summation** or **summation variable**.

Theorem $\sum_{k=1}^{n} c = nc$

Theorem If n is any positive integer and $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_n\}$ are sets of real numbers, then

(i) $\sum_{k=1}^{n} (a_k \pm b_k) = \sum_{k=1}^{n} a_k \pm \sum_{k=1}^{n} b_k$

(ii) $\sum_{k=1}^{n} ca_k = c \sum_{k=1}^{n} a_k$

Theorem **(i)** $\sum_{k=1}^{n} k = 1 + 2 + \ldots + n = \frac{n(n+1)}{2}$

(ii) $\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$

(iii) $\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + \ldots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$

Definition Let *f* be continuous and nonnegative on $[a,b]$. Let *A* be a real number, and let $f(u_k)$ be the minimum of *f* on $[x_{k-1}, x_k]$. The notation

$$A = \lim_{\Delta x \to 0} \sum_{k=1}^{n} f(u_k)\Delta x$$

means that for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $0 < \Delta x < \delta$, then

$$A - \sum_{k=1}^{n} f(u_k)\Delta x < \varepsilon .$$

A is called the **area under the graph of *f* from *a* to *b***. *A* is obtained by using **inscribed rectangular polygons.**

Similarly, *A* can be defined using **circumscribed rectangular polygons**. In this case let $f(v_k)$ be the maximum value of *f* on $[x_{k-1}, x_k]$.

Definition Let *f* be defined on a closed interval $[a,b]$, and let *P* be a partition of $[a,b]$. A **Riemann sum** of *f* for *P* is any expression of the form

$$R_P = \sum_{k=1}^{n} f(w_k)\Delta x_k ,$$

where w_k is in $[x_{k-1}, x_k]$ and $k = 1, 2, \ldots, n$.

Definition Let *f* be defined on a closed interval $[a,b]$, and let *L* be a real number. The statement

$$\lim_{\|P\| \to 0} \sum_k f(w_k)\Delta x_k = L$$

means that for every $\varepsilon > 0$ there is a $\delta > 0$ such that if *P* is a partition of $[a,b]$ with $\|P\| < \delta$, then

$$\left| \sum_k f(w_k)\Delta x_k - L \right| < \varepsilon$$

for any choice of numbers w_k in the subintervals $[x_{k-1}, x_k]$ of *P*.

The number *L* is a **limit of (Riemann) sums**.

Definition Let *f* be defined on a closed interval $[a,b]$. The **definite integral of *f* from *a* to *b***, is

$$\int_a^b f(x)dx = \lim_{\|P\| \to 0} \sum_k f(w_k)\Delta x_k ,$$

provided the limit exists. If the limit exists, we say that *f* is **integrable** on $[a,b]$ and that the definite integral **exists**. The process of finding the limit is called **evaluating the integral.** Note: The value of a definite integral is a *real number*. The numbers *a* and *b* are the **limits of integration**, *a* being the **lower limit** and *b* the **upper limit**.

Definition If $c > d$, then $\int_c^d f(x)dx = -\int_d^c f(x)dx$.

Definition If $f(a)$ exists, then $\int_a^a f(x)dx = 0$.

Theorem If f is integrable and $f(x) \geq 0$ for every x in $[a,b]$, then the area A of the region under the graph of f from a to b is

$$A = \int_a^b f(x)dx .$$

Theorem If f is continuous on $[a,b]$, then f is integrable on $[a,b]$.

Properties of the Definite Integral

1. If c is a real number, then

$$\int_a^b c\,dx = c(b-a) .$$

2. If f is integrable on $[a,b]$ and c is any real number, then cf is integrable on $[a,b]$ and

$$\int_a^b c\,f(x)dx = c\int_a^b f(x)dx .$$

3. If f and g are integrable on $[a,b]$, then $f+g$ and $f-g$ are integrable on $[a,b]$ and

$$\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx .$$

4. If $a<c<b$ and if f is integrable on both $[a,c]$ and $[c,b]$, then f is integrable on $[a,b]$ and

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx .$$

5. If f is integrable on a closed interval and *a, b,* and c are any three numbers in the interval, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx .$$

6. If f is integrable on $[a,b]$ and $f(x) \geq 0$ for every x in $[a,b]$, then

$$\int_a^b f(x)dx \geq 0 .$$

7. If f and g are integrable on $[a,b]$ and $f(x) \geq g(x)$ for every x in $[a,b]$, then

$$\int_a^b f(x)dx \geq \int_a^b g(x)dx .$$

Mean Value Theorem for Definite Integrals

If f is continuous on a closed interval $[a,b]$, then there is a number z in the open interval (a,b) such that

$$\int_a^b f(x)dx = f(z)(b-a) .$$

Definition Let f be continuous on $[a,b]$. The **average value** f_{av} of f on $[a,b]$ is

$$f_{av} = \frac{1}{b-a}\int_a^b f(x)dx$$

Fundamental Theorem of Calculus

Suppose f is continuous on a closed interval $[a,b]$.

Part I If the function G is defined by

$$G(x) = \int_a^x f(t)dt$$

for every x in $[a,b]$, then G is an antiderivative of f on $[a,b]$, i.e., $G'(x) = f(x)$.

Part II If F is any antiderivative of f on $[a,b]$, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Corollary If f is continuous on $[a,b]$ and F is any antiderivative of f, then

$$\int_a^b f(x)dx = F(x)\Big]_a^b = F(b) - F(a).$$

Theorem $\int_a^b f(x)dx = \left[\int f(x)dx\right]_a^b$

A definite integral can be evaluated by evaluating the corresponding indefinite integral.

Theorem If $u = g(x)$, then $\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$.

Theorem Let f be continuous on $[-a,a]$.

(i) If f is an even function,

$$\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx.$$

(ii) If f is an odd function,

$$\int_{-a}^a f(x)dx = 0.$$

Theorem Let f be continuous on $[a,b]$. If $a \le c \le b$, then for every x in $[a,b]$,

$$D_x\int_c^x f(t)dt = f(x).$$

Trapezoidal Rule

Let f be continuous on $[a,b]$. If a regular partition of $[a,b]$ is determined by $a = x_0, x_1, x_2, \ldots, x_n = b$, then

$$\int_a^b f(x)dx \approx \frac{b-a}{2n}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n)\right].$$

Simpson's Rule

Let f be continuous on $[a,b]$, and let n be an even integer. If a regular partition of $[a,b]$ is determined by $a = x_0, x_1, x_2, \ldots, x_n = b$, then

$$\int_a^b f(x)dx \approx \frac{b-a}{3n}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \ldots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right].$$

Integration Formulas

1. $\int u^n du = \dfrac{u^{n+1}}{n+1} + C, n \neq -1$
2. $\int \sin u \, du = -\cos u + C$
3. $\int \cos u \, du = \sin u + C$
4. $\int \sec^2 u \, du = \tan u \, du$
5. $\int \csc^2 u \, du = -\cot u + C$
6. $\int \sec u \tan u \, du = \sec u + C$
7. $\int \csc u \cot u \, du = -\csc u + C$

Strategy for integration:

Check if the integrand looks like one of these, or looks like a sum/difference/constant multiple of these. Often, a *u* – substitution is necessary to reduce the problem to a recognizable form. Algebraic techniques like multiplying, dividing, and factoring may also be used first; trig ids

Applications of Indefinite Integration

1. Solving **differential equations** subject to **initial conditions**
for example, given $f''(x)$ and conditions on f' and f
find *f*.
This is the general case of the next 3 applications.

2. Finding an **equation of a curve** given the slope of its tangent line at any point.

3. Application to **rectilinear motion**: finding the position function given either the velocity or acceleration function

4. **Economic applications**: given marginal revenue, marginal cost, or marginal profit, find the revenue function, cost function, or profit function, respectively

Note: The procedure for solving the above applications is the same, namely,
Step1. Integrate the given function.
Step2. Substitute the initial condition(s) to get the constant(s).

Definite Integration and Fundamental Theorem of Calculus II

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b = F(b) - F(a)$$

This means
Step1. Evaluate the definite integral, find an antiderivative *F* using the integration formulas.
Step2. Subtract the value of *F* at the lower limit a from the value of F at the upper limit *b.*
The answer is a **number** unlike the answer for indefinite integration, which is a **family of antiderivatives (functions)**.

Differentiation and integration are inverse processes, thus:

$$D_x \int f(x)dx = f(x) \quad \text{and} \quad \int D_x\left(f(x)\right)dx = f(x) + C$$

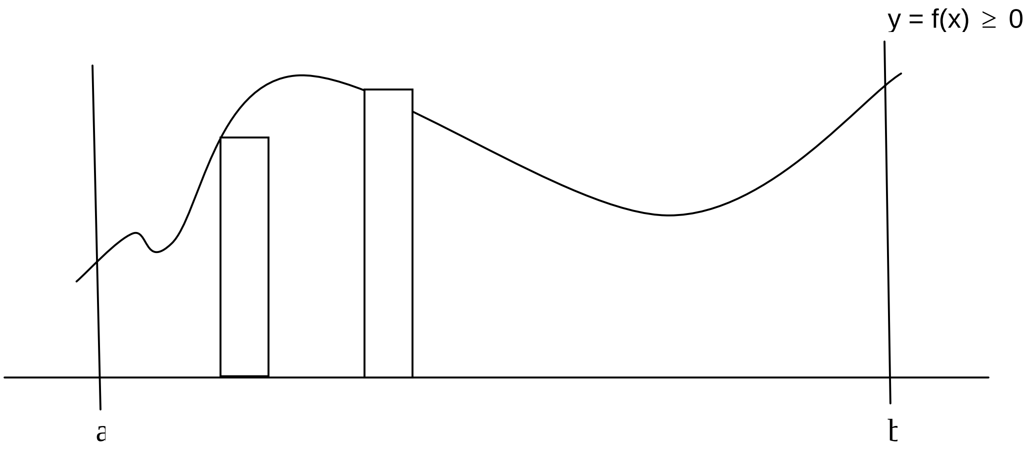
Differentiation and FTC I

$$D_x \int_a^x f(t)dt = f(x), \ a \text{ constant}$$

$$D_x \int_a^{g(x)} f(t)dt = f(g(x))g'(x), \ a \text{ constant}$$

$$D_x \int_{g(x)}^{h(x)} f(t)dt = f(h(x))h'(x) - f(g(x))g'(x)$$

Application of the Definite Integral to Area



Problem: To find the area under the curve, above the x – axis, and between the lines $x = a$ and $x = b$.

1. Partition $\left[a,b\right]$ into n subintervals of equal length $\Delta x = \dfrac{b-a}{n}$ as follows:

$$x_0 = a,\ x_1 = a + \Delta x,\ x_2 = a + 2\Delta x,\ \ldots,\ x_{k-1} = a + (k-1)\Delta x,\ x_k = a + k\Delta x,\ \ldots,\ x_n = b$$

2. For **inscribed rectangles** (see the first rectangle),

a. the area of the kth rectangle is

area of kth = (length)(width)

= $f(u_k)\Delta x$, where $f(u_k) =$ **min** of f on the subinterval

b. the area of the **inscribed polygon** is

A_{IP} = sum of the areas of the individual rectangles

$$= \sum_{k=1}^{n} f(u_k)\Delta x$$

If a specific number n is given (to get an approximation to area), stop here; otherwise to get the actual area go to (c).

c. First express A_{IP} in terms of n only using one of the following formulas:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2},\quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6},\quad \sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2$$

then evaluate the limit as $n \to \infty$

3. For **circumscribed rectangles** (see the second rectangle), change

area of kth = (length)(width)

= $f(v_k)\Delta x$, where $f(v_k) =$ **max** of f on the subinterval

then follow the same procedure as in (2).

Note. The max or min of f could occur anywhere but for simplicity they usually occur at either the left endpoint or the right endpoint of the interval. If you are using the left endpoint, then use

$$f(x_{k-1}) = f\left[a + (k-1)\Delta x\right] = f\left[a + (k-1)\frac{b-a}{n}\right].$$

For the right endpoint, use

$$f(x_k) = f\left[a + k\,\Delta x\right] = f\left[a + k\,\frac{b-a}{n}\right]$$

4. Trapezoidal Rule

Use trapezoids instead of rectangles. In this case a specific *n* will always be given.

$$\int_a^b f(x)dx \approx \frac{b-a}{2n}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n)\right].$$

5. Simpson's Rule

Use portions of parabolas and get the area under the region. A specific *n* is given and *n* has to be **even**.

$$\int_a^b f(x)dx \approx \frac{b-a}{3n}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \ldots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right].$$

Mean Value Theorem for Definite Integrals

If *f* is continuous on a closed interval $\left[a,b\right]$, then there is a number *z* in the open interval $\left(a,b\right)$ such that

$$\int_a^b f(x)dx = f(z)(b-a).$$

or

$$f(z) = \frac{1}{b-a}\int_a^b f(x)dx$$

where

$\frac{1}{b-a}\int_a^b f(x)dx$ is the **average value** f_{av} of *f* on $\left[a,b\right]$