Math 265 Third Exam Review Notes

Extrema of Functions

Definitions Let *f* be defined on an interval *I* and let x_1 and x_2 be numbers in *I*.

(i) *f* is **increasing** on *I* if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$.

(ii) *f* is **decreasing** on *I* if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$.

(iii) *f* is **constant** on *I* if $f(x_1) = f(x_2)$ for every x_1 and x_2.

Definitions Let *f* be defined on a set *S* of real numbers, and let *c* be a number in *S.*

(i) $f(c)$ is the **maximum value** of *f* on *S* if $f(x) \le f(c)$ for every *x* in *S*.

(ii) $f(c)$ is the **minimum value** of *f* on *S* if $f(x) \ge f(c)$ for every *x* in *S*.

(iii) $f(c)$ is an **extreme value** or **extremum** of *f* on *S* if If $f(c)$ is either (i) or (ii).

(iv) If $S = D_f$, then the max and min of *f* on *S*, if they exist, are called the **absolute maximum** and **absolute minimum** of *f*.

Extreme Value Theorem

If *f* is continuous on a closed interval $[a,b]$, then *f* takes on a minimum value and a maximum value at least once in $[a,b]$.

Definitions Let *c* be a number in the domain of *f*.

(i) $f(c)$ is a **local maximum** of *f* if there exists an open interval (a,b) containing *c* such that $f(x) \le f(c)$ for every *x* in (a,b).

(ii) $f(c)$ is a **local minimum** of *f* if there exists an open interval (a,b) containing *c* such that $f(x) \ge f(c)$ for every *x* in (a,b).

(iii) $f(c)$ is a **local extremum** if it is either a local max or a local min.

Theorem *f* has a local extremum at a number *c* in an open interval
$\Rightarrow$ either $f'(c) = 0$ or $f'(c)$ DNE .

Corollary $f'(c)$ exists and $f'(c) \ne 0 \Rightarrow f(c)$ is not a local extremum of *f*.

Theorem *f* is continuous on a closed interval $[a,b]$ and has its maximum or minimum at a number *c* in the open interval $(a,b) \Rightarrow f'(c) = 0$ or $f'(c)$ DNE.

Definition A number *c* in the domain of *f* is called a **critical number** of *f* if $f'(c) = 0$ or $f'(c)$ DNE.

Guidelines for finding the extrema of a function on a closed interval $[a,b]$**:**

1. Find all the critical numbers of *f* in (a,b).
2. Calculate $f(c)$ for each c.n. found in #1.
3. Calculate $f(a)$ and $f(b)$.
4. Compare the values.

Rolle's Theorem

If (i) f is continuous on a closed interval $[a,b]$,

(ii) f is differentiable on the open interval (a,b), and

(iii) $f(a)=f(b)$,

then $f'(c)=0$ for at least one number c in (a,b).

Corollary If f is continuous on $[a,b]$ and if $f(a)=f(b)$, then f has at least one critical number in (a,b).

Mean Value Theorem

If (i) f is continuous on a closed interval $[a,b]$, and

(ii) f is differentiable on the open interval (a,b),

then there exists a number c in (a,b) such that

$$f'(c) = \frac{f(b)-f(a)}{b-a},$$

or, equivalently,

$$f(b)-f(a)=f'(c)(b-a).$$

First Derivative Test

Let c be a c.n. for f, and let f be continuous at c and differentiable on an open interval I containing c, except possibly at c itself. For x in I

(i) $f'(x)>0$ for $x<c$ and $f'(x)<0$ for $x>c \Rightarrow f(c)$ is a local max of f **.**

(ii) $f'(x)<0$ for $x<c$ and $f'(x)>0$ for $x>c \Rightarrow f(c)$ is a local min of f **.**

(iii) $f'(x)>0$ or $f'(x)<0$ for all x in $I \Rightarrow f(c)$ is not a local extremum of f.

Definition Let f be differentiable on an open interval I. The graph of f is

(i) **concave upward** on I if f' is increasing on I

(ii) **concave downward** on I if f' is decreasing on I

Test for Concavity

If f'' exists on an open interval I, then

(i) $f''(x)>0$ on $I \Rightarrow$ graph of f is concave upward on I

(ii) $f''(x)<0$ on $I \Rightarrow$ graph of f is concave downward on I

Definition A point $(c,f(c))$ on the graph of f is a **point of inflection** if

(i) f is continuous at c, and

(ii) the concavity changes from the left and from the right of c

Second Derivative Test

Suppose f is differentiable on an open interval I containing c and that $f'(c)=0$.

(i) $f''(c)<0 \Rightarrow f(c)$ is a local max of f

(ii) $f''(c)>0 \Rightarrow f(c)$ is a local min of f

<u>Guidelines for graph sketching:</u>

Find/analyze the following:

1. domain
2. continuity
3. intercepts
4. symmetry
5. critical numbers and local extrema; intervals where the function is increasing/decreasing
6. concavity and points of inflection
7. asymptotes

<u>Strategy for Solving Optimization Problems</u>

1. Read the problem.

2. Draw a picture.

3. Introduce variables and write every relation in the picture.

4. Identify the quantity you want to maximize or minimize and write it as a function of *one variable only*.

5. Differentiate the function in #4 with respect to the one independent variable and solve for critical numbers.

6. Test the critical numbers and endpoints. Use FDT or SDT to classify critical numbers.

Rectilinear Motion

<u>Definition</u> Let $s(t)$ be the coordinate of a point *P* on a coordinate line *l* at time *t*.

(i) The **velocity** of *P* is $v(t) = s'(t)$.

(ii) The **speed** of *P* is $|v(t)|$.

(iii) The **acceleration** of *P* is $a(t) = v'(t) = s''(t)$.

Oscillations

<u>Definition</u> A point *P* moving on a coordinate line *l* is in **simple harmonic motion** if its distance $s(t)$ from the origin at time *t* is given by either

$$s(t) = k\cos(\omega t + b) \text{ or } s(t) = k\sin(\omega t + b),$$

where $k, \omega,$ and b are constants, with $\omega > 0$.

amplitude = $|k|$ maximum displacement of the point from the origin

period = $2\pi/\omega$ time required for one complete oscillation

frequency = $\omega/2\pi$ number of oscillations per unit of time

Economic Applications

If *x* is the number of units of a commodity, define

Cost function	$C(x)$ = cost of producing *x* units
Average Cost function	$c(x) = \dfrac{C(x)}{x}$ = average cost of producing one unit
Revenue function	$R(x)$ = revenue received for selling *x* units
Profit function	$P(x) = R(x) - C(x)$ = profit in selling *x* units
marginal cost	C'
marginal average cost	c'
marginal revenue	R'
marginal profit	P'

Newton's Method

Let *f* be a differentiable function, and let *r* be a real zero of *f*.

If x_n is an approximation of *r*, then the next approximation x_{n+1} is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

provided $f'(x_n) \neq 0$.