Math 265 Second Exam Review Notes

The Derivative

Definition The **slope m_a of the tangent line** to the graph of a function *f* at $P(a, f(a))$ is

$$m_a = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

provided the limit exists.

Application to rectilinear motion

- object travels along a straight line

Definition The **average velocity** v_{av} of an object that travels a distance *d* in time *t* is

$$v_{av} = \frac{d}{t}$$

Definition Suppose a point *P* moves on a coordinate line *l* such that its coordinates at time *t* is $s(t)$.

i. The **average velocity** is

$$v_{av} = \frac{s(a+h) - s(a)}{h}.$$

ii. The **(instantaneous) velocity** v_a of *P* at time *a* is

$$v_a = \lim_{h \to 0} \frac{s(a+h) - s(a)}{h}.$$

Definition Let $y = f(x)$, where *f* is defined on an open interval containing *a.*

i. The **average rate of change** of $y = f(x)$ wrt *x* on the interval $[a, a+h]$ is

$$y_{av} = \frac{f(a+h) - f(a)}{h}.$$

ii. The **instantaneous rate of change** of *y* wrt *x* at *a* is

$$y_a = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

provided the limit exists.

Definition The **derivative** of a function *f* is the function f' defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists.

Alternative Definition $f'(a) = \lim_{x \to a} \dfrac{f(x) - f(a)}{x - a}$

Applications of the Derivative

(i) **tangent line**

$f'(a)$ **=** slope of the tangent line to the graph of $y = f(x)$ at the point $(a, f(a))$

(ii) **rate of change**

If $y = f(x)$, the instantaneous rate of change of *y* wrt *x* at a is $f'(a)$

Def. A function is **differentiable on an open interval** (a,b) if $f'(x)$ exists for every *x* in (a,b).

Definition A function f is **differentiable on a closed interval** $[a,b]$ if f is differentiable on the open interval (a,b) and if the following limits exist:

$$\lim_{h\to 0^+}\frac{f(a+h)-f(a)}{h} \text{ and } \lim_{h\to 0^-}\frac{f(b+h)-f(b)}{h}$$

These limits are called the **right-hand derivative** and **left-hand derivative** of f at a and b, respectively.

Def. The graph of f has a **corner** at $P(a,f(a))$ if f is continuous at a and
(i) the right-hand and left-hand derivatives at a exist and are unequal, OR
(ii) if one of these derivatives exists at a and $|f'(x)|\to\infty$ as $x\to a^-$ or $x\to a^+$

Definition The graph of a function f has a **vertical tangent line** $x=a$ at the point $P(a,f(a))$ if f is continuous at a and if $\lim_{x\to a}|f'(x)|=\infty$.

Definition A point $P(a,f(a))$ on the graph of a function f is a **cusp** if f is continuous at a and if the following conditions hold:
(i) $f'(x)\to\infty$ as x approaches a from one side
(ii) $f'(x)\to-\infty$ as x approaches a from the other side

Theorem If a function f is differentiable at a, then f is continuous at a.

Note The converse is not necessarily true.

Notations for the Derivative

$$f'(x),\ f''(x),\ f'''(x),\ f^{(4)}(x),\ldots,f^{(n)}(x)$$
$$D_x y,\ D_x^2 y,\ D_x^3 y,\ D_x^4 y,\ \ldots,\ D_x^n y$$
$$y',\ y'',\ y''',\ y^{(4)},\ \ldots,\ y^{(n)}$$
$$\frac{dy}{dx},\ \frac{d^2y}{dx^2},\ \frac{d^3y}{dx^3},\ \frac{d^4y}{dx^4},\ \ldots,\frac{d^ny}{dx^n}$$

Differentiation Rules

1. $D_x c = 0$ — The derivative of a constant is 0.

2. $D_x(mx+b)=m$

3. $D_x\left(x^n\right)=nx^{n-1}$ — Power Rule

4. $D_x\left[cf(x)\right]=c\,D_x f(x)$ — The derivative of a constant times a function is the constant times the derivative of the function.

5. $D_x\left[f(x)\pm g(x)\right]=D_x f(x)\pm D_x g(x)$ — The derivative of the sum/difference is the sum of the derivatives.

6. $D_x\left[f(x)g(x)\right]=f(x)D_x g(x)+g(x)D_x f(x)$ — Product Rule: The derivative of a product is the first factor times the derivative of the second, plus the second factor times the derivative of the first.

7. $D_x\left[\dfrac{f(x)}{g(x)}\right]=\dfrac{g(x)D_xf(x)-f(x)D_xg(x)}{[g(x)]^2}$

Quotient Rule: The derivative of a quotient is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, ALL OVER denominator squared.

8. $D_x\left[\dfrac{1}{g(x)}\right]=-\dfrac{D_xg(x)}{[g(x)]^2}$

Reciprocal Rule

9. $D_x[f(g(x)]=f'(g(x))g'(x)$

Chain Rule

10. $D_x[g(x)]^n=n[g(x)]^{n-1}D_xg(x)$

Generalized Power Rule

Important Trigonometric Limits

1. $\lim\limits_{\theta\to 0}\sin\theta=0$

2. $\lim\limits_{\theta\to 0}\cos\theta=1$

3. $\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta}=1$

4. $\lim\limits_{\theta\to 0}\dfrac{1-\cos\theta}{\theta}=0$

Derivatives of the Trigonometric Functions

$D_x\sin x=\cos x$ $\qquad D_x\cos x=-\sin x$

$D_x\tan x=\sec^2 x$ $\qquad D_x\cot x=-\csc^2 x$

$D_x\sec x=\sec x\tan x$ $\qquad D_x\csc x=-\csc x\cot x$

Def If f is a differentiable function, then the **normal line** at a point $P(a,f(a))$ on the graph of f is the line through P that is perpendicular to the tangent line.

Increments and Differentials

increment of x $\quad \Delta x=x_1-x_0$

increment of y $\quad \Delta y=f(x_1)-f(x_0)=f(x_0+\Delta x)-f(x_0)$

Increment Definition of the Derivative

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}$$

Definition Let $y=f(x)$ be differentiable, and let Δx be an increment of x.

(i) The **differential *dx*** of the independent variable x is $dx=\Delta x$.

(ii) The **differential *dy*** of the dependent variable y is

$$dy=f'(x)\,\Delta x=f'(x)\,dx=(D_xy)dx.$$

Linear Approximation Formula

If $y=f(x)$ is differentiable, and Δx is an increment of x, then

$$f(x+\Delta x)\approx f(x)+dy$$

<u>Definition</u> If w denotes a measurement with a maximum error Δw , then

(i) **average error** = $\dfrac{\Delta w}{w}$

(ii) **percentage error** = (average error) $\times$ (100%)

Implicit Differentiation

- method used to find $D_x y$ when y is not defined as an **explicit function** of x

Related Rates

<u>General Strategy for Solving Related Rates Problems</u>

1. Read the problem.

2. Sketch a picture or diagram. Label the quantities that do not change with time with their given constant values; use variables for quantities that change with time.

3. Express the rates of change of quantities that change with time as derivatives wrt time t.

4. Write a **general equation** that relates the variables. The equation must be **true for any time** t.

5. Differentiate the equation obtained in #4 implicitly wrt time t to obtain a general relationship between the rates of change.

6. Substitute the values and rates that hold **at the particular instant of time** under consideration.