Math 265 First Exam Review Notes

Limits of Functions

Notation $\lim_{x\to a} f(x) = L \Leftrightarrow f(x)\to L$ as $x\to a$

Theorem (one-sided limits and limits) $\lim_{x\to a} f(x) = L \Leftrightarrow \lim_{x\to a^-} f(x) = L = \lim_{x\to a^+}$

Definition of Limit Let f be defined on an open interval containing a, except possibly at a itself, and let L be a real number. The statement

$$\lim_{x\to a} f(x) = L$$

means that for every $\varepsilon>0$, there is a $\delta>0$ such that

$$0<|x-a|<\delta \Rightarrow |f(x)-L|<\varepsilon$$

Restatement of the Definition $\lim_{x\to a} f(x) = L$

means that for every $\varepsilon>0$, there is a $\delta>0$ such that

$x\in(a-\delta, a+\delta)$ and $x\neq a \Rightarrow f(x)\in(L-\varepsilon, L+\varepsilon)$

Steps in showing that $\lim_{x\to a} f(x) = L$ using the definition:

Step1. Consider any $\varepsilon>0$.

Step2. Show that there is a $\delta>0$ such that if x has δ-tolerance at a, then $f(x)$ has ε-tolerance at L.

Note: δ is not unique, for if a specific δ is found, then any smaller positive number δ_1 will also satisy the requirements.

Theorem If $\lim_{x\to a} f(x) = L > 0$, then there is an open interval $(a-\delta, a+\delta)$ containing a such that $f(x)>0$ for every $x\in(a-\delta, a+\delta)$, except possibly at $x=a$.

Techniques for Finding Limits

1. $\lim_{x\to a} c = c$, c constant

2. $\lim_{x\to a} x = a$

If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, and both exist, then

3. $\lim_{x\to a}[f(x)\pm g(x)] = L\pm M$

4. $\lim_{x\to a}[f(x)\cdot g(x)] = LM$

5. $\lim_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{L}{M}$, $M\neq 0$

6. $\lim_{x\to a} cf(x) = cL$

7. If n is a positive integer, then

i. $\lim_{x\to a} x^n = a^n$

ii. $\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$, provided $\lim_{x\to a} f(x)$ exists

8. If *f* is a polynomial function and *a* is a real number, then $\lim_{x \to a} f(x) = f(a)$.

9. If *q* is a rational function and *a* is in the domain of *q*, then $\lim_{x \to a} q(x) = q(a)$.

10. If $a > 0$ and *n* is a positive integer, or
if $a \le 0$ and *n* is a negative integer, then

$$\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$$

11. If $\lim_{x \to a} f(x)$ exists, then $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$, provided either
n is an odd positive integer or
n is an even positive integer and $\lim_{x \to a} f(x) > 0$.

12. Sandwich Theorem
Suppose $f(x) \le h(x) \le g(x)$ for every *x* in the open interval containing *a*, except possibly at *a*.

$$\lim_{x \to a} f(x) = L = \lim_{x \to a} g(x) \Rightarrow \lim_{x \to a} h(x) = L$$

Limits at Infinity

Definition (i) $\lim_{x \to \infty} f(x) = L$ means that for every $\varepsilon > 0$, there is a number $M > 0$ such that

$$x > M \Rightarrow |f(x) - L| < \varepsilon .$$

(ii) $\lim_{x \to -\infty} f(x) = L$ means that for every $\varepsilon > 0$, there is a number $N < 0$ such that

$$x < N \Rightarrow |f(x) - L| < \varepsilon .$$

Theorem If *k* is a positive rational number and *c* is any real number, then

$$\lim_{x \to \pm\infty} \frac{c}{x^k} = 0 .$$

This theorem is useful for finding limits of rational functions.
Technique: First divide the numerator and denominator of the function by x^n, where *n* is the highest power of *x* that appears in the denominator, and then use limit theorems.

Infinite Limits

Definition $\lim_{x \to a} f(x) = \infty$ means that for every $M > 0$, there is a $\delta > 0$ such that

$$0 < |x - a| < \delta \Rightarrow f(x) > M .$$

Continuous Functions

Definition A function *f* is **continuous** at a number *c* if the following conditions are satisfied:

i. $f(c)$ is defined

ii. $\lim_{x \to c} f(x)$ exists

iii. $\lim_{x \to c} f(x) = f(c)$

If one (or more) of the three conditions is not satisfied, then *f* is said to be **discontinuous** at *c*, or that *f* **has a discontinuity** at *c*.

Types of discontinuity

removable discontinuity	(ii) is satisfied but (i) and/or (iii) not
jump discontinuity	(ii) is not satisfied (left-hand limit $\ne$ right-hand limit)
infinite discontinuity	$\lim_{x \to c} f(x) = \pm\infty$

Definition If f is continuous at every number in an open interval (a,b), then f is said to be **continuous on the interval (*a,b*)**.

Definition Let a function f be defined on a closed interval $[a,b]$. f is **continuous on [*a,b*],** if It is continuous on (a,b) and if,

$$\lim_{x\to a^+} f(x) = f(a) \text{ and } \lim_{x\to a^-} f(x) = f(b) \text{, i.e.}$$

f is **continuous from the right at *a*** and f is **continuous from the left at *b***.

Theorems

1. A polynomial function f is continuous at every real number c.

2. A rational function $q = f/g$ is continuous at every real number c except for the numbers c such that $g(c) = 0$.

3. If f and g are continuous at c, then the following are also continuous at c:

- **i.** $f + g$
- **ii.** $f - g$
- **iii.** fg
- **iv.** f/g, provided $g(c) \neq 0$

4. If $\lim_{x\to c} g(x) = b$ and if f is continuous at b, then

$$\lim_{x\to c} f(g(x)) = f(b) = f\left(\lim_{x\to c} g(x)\right).$$

5. If g is continuous at c and if f is continuous at $b = g(c)$, then

- **i.** $\lim_{x\to c} f(g(x)) = f\left(\lim_{x\to c} g(x)\right) = f(g(c))$
- **ii.** The composite function $f \circ g$ is continuous at c.

6. Intermediate Value Theorem

If f is continuous on a closed interval $[a,b]$ and if w is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a,b]$ such that $f(c) = w$.

7. If f is continuous and has no zeros on an interval, then either $f(x) > 0$ or $f(x) < 0$ for every x in the interval.