Math 270 Linear Algebra

Chapter 6 Linear Transformations and Matrices

6.2 Kernel and Range of a Linear Transformation

Definition $L: V \to W$ is a **one-to-one** linear transformation if it is a 1-1 function, i.e.

$$L(\vec{v}_1) = L(\vec{v}_2) \Rightarrow \vec{v}_1 = \vec{v}_2$$

or equivalently,

$$\vec{v}_1 \neq \vec{v}_2 \Rightarrow L(\vec{v}_1) \neq L(\vec{v}_2).$$

Exercise #4 $L: \mathbb{R}_2 \to \mathbb{R}_3$

$$\begin{bmatrix} u_1 & u_2 \end{bmatrix} \to \begin{bmatrix} u_1 & u_1 + u_2 & u_2 \end{bmatrix}$$

Is L one-to-one?

Solution:

Let $\vec{u} = \begin{bmatrix} u_1 & u_2 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$

$$L(\vec{u}) = L(\vec{v}) \Rightarrow \begin{bmatrix} u_1 & u_1 + u_2 & u_2 \end{bmatrix} = \begin{bmatrix} v_1 & v_1 + v_2 & v_2 \end{bmatrix}$$

$$\Rightarrow \begin{cases} u_1 = v_1 \\ u_1 + u_2 = v_1 + v_2 \\ u_2 = v_2 \end{cases} \Rightarrow \vec{u} = \vec{v}$$

Thus, L is 1-1.

Exercise #2 $L: \mathbb{R}^2 \to \mathbb{R}^2$

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Is L 1-1?

Solution:

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\Rightarrow \begin{cases} u_1 + 2u_2 = v_1 + 2v_2 \\ 2u_1 + 4u_2 = 2v_1 + 4v_2 \end{cases}$$

same equation!

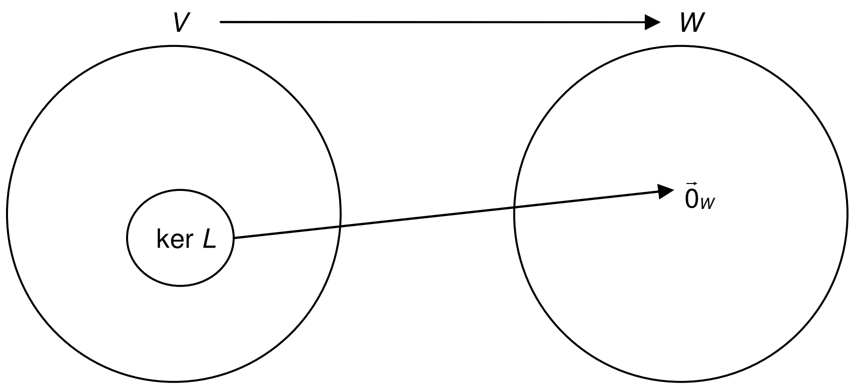
L is NOT $1 - 1$ since for example

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ but } \begin{bmatrix} 0 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

To develop a more efficient way of determining whether L is 1 – 1 or not:

Definition Let $L: V \to W$ be a linear transformation. The **kernel** of L, ker L, is the subset of V consisting of all elements $\vec{v} \in V$ s.t. $L(\vec{v}) = \vec{0}_W$.

$$\ker\ L = \left\{ \vec{v} \in V \middle| L(\vec{v}) = \vec{0}_W \right\}$$



Note: $\vec{0}_V \in \ker L \Rightarrow \ker L \neq \{\ \}$

Exercise #2 $L: \mathbb{R}^2 \to \mathbb{R}^2$

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Find ker L.

Solution:

$$L\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} u_1 + 2u_2 = 0 \\ 2u_1 + 4u_2 = 0 \end{cases} \Rightarrow u_1 = -2u_2$$

$$\Rightarrow \ker L = \left\{ \begin{bmatrix} -2r \\ r \end{bmatrix},\ r \text{ real} \right\}$$

For example, $\begin{bmatrix} -2 \\ 1 \end{bmatrix} \in \ker L$, $\begin{bmatrix} 6 \\ -3 \end{bmatrix} \in \ker L$

Exercise #4 $L: \mathbb{R}_2 \to \mathbb{R}_3$

$$\begin{bmatrix} u_1 & u_2 \end{bmatrix} \to \begin{bmatrix} u_1 & u_1 + u_2 & u_2 \end{bmatrix}$$

Find ker L.

Solution:

$$L\left(\begin{bmatrix} u_1 & u_2 \end{bmatrix}\right) = \begin{bmatrix} u_1 & u_1 + u_2 & u_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} u_1 = 0 \\ u_1 + u_2 = 0 \Rightarrow u_1 = u_2 = 0 \\ u_2 = 0 \end{cases}$$

$$\Rightarrow \ker L = \left\{ \begin{bmatrix} 0 & 0 \end{bmatrix} \right\}$$

Theorem Let $L: V \to W$ be a linear transformation …

(a) $\ker L$ is a subspace of V.

(b) L is 1-1 $\Leftrightarrow \ker L = \{\vec{0}_V\}$.

Proof:

(a) $\vec{v}, \vec{w} \in \ker L \Rightarrow L(\vec{v}) = \vec{0}_W$ and $L(\vec{w}) = \vec{0}_W$

$L(\vec{v} + \vec{w}) = L(\vec{v}) + L(\vec{w}) = \vec{0}_W + \vec{0}_W = \vec{0}_W \Rightarrow \vec{v} + \vec{w} \in \ker L$

For c real, $L(c\vec{v}) = cL(\vec{v}) = c\vec{0}_W = \vec{0}_W \Rightarrow c\vec{v} \in \ker L$

Thus, $\ker L$ is a subspace of V.

(b) ("$\Rightarrow$") Let L be $1-1$. To show $\ker L = \{\vec{0}_V\}$.

$\vec{v} \in \ker L \Rightarrow L(\vec{v}) = \vec{0}_W = L(\vec{0}_V)$

$\Rightarrow \vec{v} = \vec{0}_V$

$\Rightarrow \ker L = \{\vec{0}_V\}$

("$\Leftarrow$") Let $\ker L = \{\vec{0}_V\}$. To show L is $1-1$.

Let $L(\vec{v}_1) = L(\vec{v}_2)$ for $\vec{v}_1,\ \vec{v}_2 \in V$

$\Rightarrow L(\vec{v}_1) - L(\vec{v}_2) = \vec{0}_W$

$\Rightarrow L(\vec{v}_1 - \vec{v}_2) = \vec{0}_W$

$\Rightarrow \vec{v}_1 - \vec{v}_2 = \vec{0}_V$

$\Rightarrow \vec{v}_1 = \vec{v}_2$

$\Rightarrow L$ is 1-1

Restatement of (b): L is 1-1 $\Leftrightarrow \dim\ \ker L = 0$.

Corollary If $L(\vec{x}) = \vec{b}$ and $L(\vec{y}) = \vec{b}$ then $\vec{x} - \vec{y} \in \ker L$.

For Exercise #2, since $\ker L \neq \{\vec{0}_V\}$, L is not 1-1.

For Exercise #4, since $\ker L = \{\vec{0}_V\}$, L is 1-1.

Exercise #6 Let $L : P_2 \to P_2$

$$L(p(t)) = t^2 p'(t)$$

a) Find $\ker L$.
b) Find dim $\ker L$ and a basis for $\ker L$.
c) Is L 1-1?

Solution:
a)

$$L(p(t)) = t^2 p'(t) = 0 \Rightarrow t^2 = 0 \text{ or } p'(t) = 0$$
$$\Rightarrow \quad p(t) = c$$
$$\Rightarrow \ker L = \{p \in P_2 \mid p(t) = c\}$$

b)

$$p(t) = c = c \cdot 1 \Rightarrow \text{basis for } \ker L = \{p(t) \in P_2 \mid p(t) = 1\}$$
$$\Rightarrow \dim \ \ker L = 1$$

c) L is not 1-1 since dim ker $L \neq 0$.

Definition Let $L : V \to W$. The **range** of L or the **image** of V under L, denoted by range L, consists of all those vectors $\vec{w} \in W$ that are images under L of vectors in V.

$$\text{range } L = \left\{\vec{w} \in W \,\middle|\, L(\vec{v}) = \vec{w} \text{ for some } \vec{v} \in V\right\}.$$

L is called **onto** if range $L = W$.

Theorem If $L : V \to W$ is a linear transformation of a vector space V into a vector space W, then range L is a subspace of W.

Proof:

$$\vec{w}_1, \vec{w}_2 \in \text{range } L \Rightarrow L(\vec{v}_1) = \vec{w}_1 \text{ and } L(\vec{v}_2) = \vec{w}_2 \text{ for some } \vec{v}_1, \vec{v}_2 \in V$$
$$\Rightarrow \vec{w}_1 + \vec{w}_2 = L(\vec{v}_1) + L(\vec{v}_2) = L(\underbrace{\vec{v}_1 + \vec{v}_2}_{\in V})$$
$$\Rightarrow \vec{w}_1 + \vec{w}_2 \in \text{ range } L$$

$$c \text{ real scalar, } \vec{w} \in \text{ range } L \Rightarrow L(\vec{v}) = \vec{w} \text{ for some } \vec{v} \in V$$
$$c\vec{w} = cL(\vec{v}) = L(\underbrace{c\vec{v}}_{\in V})$$
$$\Rightarrow c\vec{w} \in \text{ range } L$$

Hence, range L is a subspace of W.

Exercise #2 $L:\mathbb{R}^2 \to \mathbb{R}^2$

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

a) Is $\begin{bmatrix} 3 \\ 6 \end{bmatrix} \in$ range L?

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 2 & 4 & 6 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} u_1 = 3-2r \\ u_2 = r \end{array}$$

For ex, when $r=0,\ u_1=3,\ u_2=0$.

YES

b) Is $\begin{bmatrix} 2 \\ 3 \end{bmatrix} \in$ range L?

$$\left[\begin{array}{cc|c} 1 & 2 & 2 \\ 2 & 4 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & 0 & -1 \end{array}\right] \Rightarrow \text{NO}$$

c) Find a set of vectors spanning range L.

$$L\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_1 + 2u_2 \\ 2u_1 + 4u_2 \end{bmatrix} = u_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + u_2 \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$\Rightarrow \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right\}$ spans range L

Note: The vectors are not linearly independent.

Exercise #4 $L:\mathbb{R}_2 \to \mathbb{R}_3$

$$\begin{bmatrix} u_1 & u_2 \end{bmatrix} \to \begin{bmatrix} u_1 & u_1 + u_2 & u_2 \end{bmatrix}$$

Is L onto?

Solution:

For any $\begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix} \in \mathbb{R}_3$ can we find $\begin{bmatrix} u_1 & u_2 \end{bmatrix} \in \mathbb{R}_2$ s.t.

$$L\left(\begin{bmatrix} u_1 & u_2 \end{bmatrix}\right) = \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix}?$$

$$\begin{bmatrix} u_1 & u_1 + u_2 & u_2 \end{bmatrix} = \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix}$$

$$\Rightarrow \begin{cases} u_1 = w_1 \\ u_1 + u_2 = w_2 \Rightarrow w_1 + w_2 = w_3, \text{ a special requirement} \Rightarrow \text{NO} \\ u_2 = w_3 \end{cases}$$

Exercise #6 Let $L: P_2 \to P_2$

$$L(p(t)) = t^2 p'(t)$$

Find a basis for and dimension of range L.

Solution:

$$p(t) \in P_2 \Rightarrow p(t) = at^2 + bt + c$$

$$L(p(t)) = L(at^2 + bt + c) = t^2(2at + b) = 2at^3 + bt^2$$

$\Rightarrow \{2t^3, t^2\}$ spans range L and is a basis for range L

dim range $L = 2$

Note: For Exercise #6
dim ker L = 1, dim range L = 2
dim ker L + dim range L = 3 = dim domain L

Theorem If $L: V \to W$ is a linear transformation of an n-dimensional vector space V into a vector space W, then

$$\dim \ \ker L + \dim \ \text{range} L = \dim V.$$

Proof:

Let $k = \dim \ \ker L$

(1) If $k = n$, then $\ker L = V \Rightarrow L(\vec{v}) = \vec{0}_W \ \forall \vec{v} \in V$

$\Rightarrow$ range $L = \{\vec{0}_W\}$

$\Rightarrow$ dim range$L = 0$

(2) If $1 \le k < n$, show dim range$L = n - k$.

Let $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ be a basis for ker L. Extend this to a basis $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k, \vec{v}_{k+1}, \ldots, \vec{v}_n\}$ for V. To prove that the set $T = \{L(\vec{v}_{k+1}), L(\vec{v}_{k+2}), \ldots, L(\vec{v}_n)\}$ is a basis for range L.

(i) T spans range L:

Let $\vec{w} \in$ range$L$. Then $\vec{w} = L(\vec{v})$ for some $\vec{v} \in V$.

Since S is a basis for V, then there exist unique scalars $a_1, \ldots, a_n$ s.t.

$$\vec{v} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n.$$

$$\begin{aligned}\vec{w} = L(\vec{v}) &= L(a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n) \\ &= \underbrace{a_1L(\vec{v}_1) + a_2L(\vec{v}_2) + \cdots + a_kL(\vec{v}_k)}_{\vec{0}_W} + a_{k+1}L(\vec{v}_{k+1}) + \ldots + a_nL(\vec{v}_n) \\ &= a_{k+1}L(\vec{v}_{k+1}) + \ldots + a_nL(\vec{v}_n)\end{aligned}$$

(ii) T is linearly independent:

$$a_{k+1}L(\vec{v}_{k+1}) + a_{k+2}L(\vec{v}_{k+2}) + \ldots + a_n L(\vec{v}_n) = \vec{0}_W$$
$$\Rightarrow L(a_{k+1}\vec{v}_{k+1} + a_{k+2}\vec{v}_{k+2} + \ldots + a_n\vec{v}_n) = \vec{0}_W$$
$$\Rightarrow a_{k+1}\vec{v}_{k+1} + a_{k+2}\vec{v}_{k+2} + \ldots + a_n\vec{v}_n \in \ker L$$

Also,

$$a_{k+1}\vec{v}_{k+1} + a_{k+2}\vec{v}_{k+2} + \ldots + a_n\vec{v}_n = b_1\vec{v}_1 + b_2\vec{v}_2 + \ldots + b_k\vec{v}_k$$

for unique b_i. This implies

$$b_1\vec{v}_1 + b_2\vec{v}_2 + \ldots + b_k\vec{v}_k - a_{k+1}\vec{v}_{k+1} - a_{k+2}\vec{v}_{k+2} - \ldots - a_n\vec{v}_n = \vec{0}_V$$

and $b_1 = b_2 = \cdots = b_k = a_{k+1} = a_{k+2} = \ldots = a_n = 0$

since S is linearly independent.

(3) If $k = 0$, then ker L has no basis.
Let $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a basis for V.

Note: dim $\ker L =$ nullity of L

Exercise #10 $L : M_{22} \to M_{22}$

$$L(A) = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} A - A \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

(a) Find a basis for ker L.
(b) Find a basis for range L.

Solution:
(a) $\ker L = \{A \in M_{22} \mid L(A) = O_2\}$

$$\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} A - A \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$
$$\begin{bmatrix} a+2c & b+2d \\ a+c & b+d \end{bmatrix} - \begin{bmatrix} a+b & 2a+b \\ c+d & 2c+d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 2c-b & 2d-2a \\ a-d & b-2c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{array}{c} 2c-b=0 \\ 2d-2a=0 \\ a-d=0 \\ b-2c=0 \end{array} \Rightarrow \begin{array}{c} b=2c \\ d=a \end{array}$$

Thus, elements of ker L look like $\begin{bmatrix} a & 2c \\ c & a \end{bmatrix}$.

$$\begin{bmatrix} a & 2c \\ c & a \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$$

Hence, a basis for ker L is $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}\right\}$ and dim ker L = 2.

(b) range $L = \{B \in M_{22} \,|\, L(A) = B\}$

$$\begin{aligned} L(A) &= \begin{bmatrix} 2c-b & 2d-2a \\ a-d & b-2c \end{bmatrix} \\ &= \begin{bmatrix} 2c-b & -2(a-d) \\ a-d & -(2c-b) \end{bmatrix} \\ &= a\begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix} + b\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} + c\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} + d\begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix} \end{aligned}$$

$\Rightarrow \left\{\begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}\right\}$ spans range L

$\Rightarrow \left\{\begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\right\}$ is a basis for range L and dim range $L = 2$

Note: dim ker L + dim range L = 2 + 2 = 4 = dim M_{22}.

Corollary If $L: V \to W$ is a linear transformation of a vector space V into a vector space W and dim V = dim W, then

(a) L is 1 – 1 $\Rightarrow$ L is onto
(b) L is onto $\Rightarrow$ L is 1 – 1.

Proof: Exercise # 31

Definition $L: V \to W$ is called **<u>invertible</u>** if there exists a unique function $L^{-1}: W \to V$ s.t.

$$L \circ L^{-1} = I_W \text{ and } L^{-1} \circ L = I_V,$$

where I_V is the identity linear transformation on V, and
I_W is the identity linear transformation on W.

Theorem A linear transformation $L: V \to W$ is invertible $\Leftrightarrow$ L is 1 – 1 and onto.
Moreover, L^{-1} is a linear transformation and $\left(L^{-1}\right)^{-1} = L$.

Proof:
(" $\Leftarrow$ ") Let L be 1 – 1 and onto.
Define $H: W \to V$ as follows:

If $\vec{w} \in W$, then $\vec{w} \in L(\vec{v})$ for some unique $\vec{v} \in V$ (since L is 1 – 1 and onto).

Let $H(\vec{w}) = \vec{v}$. Then $L(H(\vec{w})) = L(\vec{v}) = \vec{w} \Rightarrow L \circ H = I_W$.

Also, $H(L(\vec{v})) = H(\vec{w}) = \vec{v} \Rightarrow H \circ L = I_V$.

Thus, H is an inverse of L.

To show H is unique:
Let $H_1 : W \to V$ be a function s.t. $L \circ H_1 = I_W$ and $H_1 \circ L = I_V$.

Then $L(H(\vec{w})) = \vec{w} = L(H_1(\vec{w}))$ for any $\vec{w} \in W$

$\Rightarrow H(\vec{w}) = H_1(\vec{w})$ since L is 1 - 1

$\Rightarrow H = H_1$.

Hence, $H = L^{-1}$.

("$\Rightarrow$") Let L be invertible. Then $L \circ L^{-1} = I_W$ and $L^{-1} \circ L = I_V$.
To show L is 1 – 1 and onto.

$L(\vec{v}_1) = L(\vec{v}_2)$ for $\vec{v}_1, \vec{v}_2 \in V$

$\Rightarrow L^{-1}(L(\vec{v}_1)) = L^{-1}(L(\vec{v}_2))$

$\Rightarrow \vec{v}_1 = \vec{v}_2$

$\Rightarrow L$ is 1 - 1

$\vec{w} \in W \Rightarrow L(L^{-1}(\vec{w})) = \vec{w}$

Let $\vec{v} = L^{-1}(\vec{w})$. Then $L(\vec{v}) = \vec{w}$ and L is onto.

Next, show L^{-1} is a linear transformation:

Let $\vec{w}_1, \vec{w}_2 \in W$ where $L(\vec{v}_1) = \vec{w}_1$ and $L(\vec{v}_2) = \vec{w}_2$ for $\vec{v}_1, \vec{v}_2 \in V$.

$L(a\vec{v}_1 + b\vec{v}_2) = aL(\vec{v}_1) + bL(\vec{v}_2) = a\vec{w}_1 + b\vec{w}_2$, a, b real

$\Rightarrow L^{-1}(a\vec{w}_1 + b\vec{w}_2) = a\vec{v}_1 + b\vec{v}_2 = aL^{-1}(\vec{w}_1) + bL^{-1}(\vec{w}_2)$

$\Rightarrow L^{-1}$ is a linear transformation

Finally, since

$$L \circ L^{-1} = I_w \text{ and } L^{-1} \circ L = I_V$$

and inverses are unique, then $(L^{-1})^{-1} = L$.

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$L : \mathbb{R}^3 \to \mathbb{R}^3$

$$L\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

To find ker L:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$\Rightarrow \ker L = \left\{\vec{0}_V\right\} \Rightarrow L$ is 1 - 1 $\Rightarrow L$ is onto $\Rightarrow L^{-1}$ exists

To find L^{-1}:

$L^{-1}\left(\vec{w}\right) = \vec{v} \Rightarrow L\left(\vec{v}\right) = \vec{w}$. Solve for $\vec{v}$.

$$L\left(\vec{v}\right) = L\left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right) = \begin{bmatrix} v_1 + v_2 + v_3 \\ 2v_1 + 2v_2 + v_3 \\ v_2 + v_3 \end{bmatrix} = \vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & w_1 \\ 2 & 2 & 1 & w_2 \\ 0 & 1 & 1 & w_3 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & w_1 \\ 0 & 0 & -1 & w_2 - 2w_1 \\ 0 & 1 & 1 & w_3 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & w_1 \\ 0 & 1 & 1 & w_3 \\ 0 & 0 & -1 & w_2 - 2w_1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & w_1 - w_3 \\ 0 & 1 & 1 & w_3 \\ 0 & 0 & -1 & w_2 - 2w_1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & w_1 - w_3 \\ 0 & 1 & 0 & w_3 + w_2 - 2w_1 \\ 0 & 0 & 1 & 2w_1 - w_2 \end{array}\right]$$

$$\Rightarrow \vec{v} = L^{-1}\left(\vec{w}\right) = \begin{bmatrix} w_1 - w_3 \\ w_3 + w_2 - 2w_1 \\ 2w_1 - w_2 \end{bmatrix}$$

We could have done the above as follows:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & -2 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & -2 & 1 & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -2 & 1 & 1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right]$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\ker L = \{\vec{0}_V\} \qquad L^{-1}(\vec{w}) = \begin{bmatrix} 1 & 0 & -1 \\ -2 & 1 & 1 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

Theorem $L: V \to W$ $1-1$ linear transformation
$\Leftrightarrow L(\text{linearly independent set } S) = \text{linearly independent set } T$

Proof:

Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ be a linearly independent set in V.

$T = \{L(\vec{v}_1), L(\vec{v}_2), \ldots, L(\vec{v}_k)\}$

$(" \Rightarrow ")$ Suppose L is $1-1$. Show T is linearly independent.

$a_1 L(\vec{v}_1) + a_2 L(\vec{v}_2) + \ldots + a_k L(\vec{v}_k) = \vec{0}_W$

$\Rightarrow L(a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_k\vec{v}_k) = \vec{0}_W$

$\Rightarrow a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_k\vec{v}_k = \vec{0}_V$ since L is 1 - 1

$\Rightarrow a_1 = a_2 = \cdots = a_k = 0$ since S is linearly independent

$\Rightarrow T$ is linearly independent

$(" \Leftarrow ")$ $\{\vec{v}\}$, $\vec{v} \neq \vec{0}$, is linearly independent in V

$\Rightarrow L(\vec{v})$ is linearly independent

$\Rightarrow L(\vec{v}) \neq \vec{0}_W$

$\Rightarrow \ker L = \{\vec{0}_V\}$

$\Rightarrow L$ is 1 - 1

Remark Let $L: V \to W$ be a linear transformation with dim V = dim W.
Then L is $1-1$ and thus invertible $\Leftrightarrow$ the image of a basis for V under L is a basis for W.
(Exercise #18)

$A\vec{x} = \vec{b}, \quad A \; n \times n$

$L : \mathbb{R}^n \to \mathbb{R}^n$

$L(\vec{x}) = A\vec{x}, \quad \vec{x} \in \mathbb{R}^n$

(1) A is nonsingular $\Rightarrow$ dim range L = rank A = n, so dim ker $L = 0$
$\Rightarrow L$ is 1 - 1 (and onto)
$\Rightarrow A\vec{x} = \vec{b}$ has a unique solution

(2) A is singular $\Rightarrow$ rank $A < n$
$\Rightarrow$ dim ker $L = n -$ rank $A > 0$
$\Rightarrow L$ is not 1 - 1 and not onto
$\Rightarrow \exists \; \vec{b} \in \mathbb{R}^n$ s.t. $A\vec{x} = \vec{b}$ has no solution

Also, $A\vec{x} = \vec{0}$ has a nontrivial solution $\vec{x}_0$.
If $A\vec{x} = \vec{b}$ has a solution $\vec{y}$ then $\vec{y} + \vec{x}_0$ is a solution to $A\vec{x} = \vec{b}$.
This implies that the solution, if it exists, is not unique.