Math 270 Linear Algebra

Chapter 6 Linear Transformations and Matrices

6.1 Definitions and Examples

Definition Let *V* and *W* be vector spaces. A function $L: V \to W$ is called a **linear transformation** of *V* into *W* if

(a) $L(\vec{u}+\vec{v}) = L(\vec{u}) + L(\vec{v})$ for $\vec{u},\ \vec{v} \in V$

(b) $L(c\vec{u}) = cL(\vec{u})$ for $\vec{u} \in V,\ c \in \mathbb{R}$

If $V = W$, then $L: V \to V$ is called a **linear operator** on *V*.

Note:

(1) + on the left is + in *V*; + on the right is + in *W*.

(2) $c\vec{u}$ on the left is scalar product in *V*; $cL(\vec{u})$ on the right is scalar product in *W*.

(3) *L* is a linear transformation $\Leftrightarrow L(a\vec{u}+b\vec{v}) = aL(\vec{u}) + bL(\vec{v})$. (Exercise 6)

Example 1 Let *A* be *m* x *n*.

Define $L: \mathbb{R}^n \to \mathbb{R}^m$ by

$$L(\vec{u}) = A\vec{u}$$

Show that *L* is a linear transformation.

Solution:

(a) For $\vec{u}, \vec{v} \in \mathbb{R}^n$

$$L(\vec{u}+\vec{v}) = A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v} = L(\vec{u}) + L(\vec{v})$$

(b) For *c* scalar, $\vec{u} \in \mathbb{R}^+$

$$L(c\vec{u}) = A(c\vec{u}) = cA(\vec{u}) = cL(\vec{u})$$

Some important matrix transformations (from Section 1.6)

1. reflection wrt the *x*-axis

$$L: \mathbb{R}^2 \to \mathbb{R}^2$$

$$L\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = \begin{bmatrix} u_1 \\ -u_2 \end{bmatrix}$$

2. projection into the *xy*-plane

$$L: \mathbb{R}^3 \to \mathbb{R}^2$$

$$L\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

3. dilation

$$L: \mathbb{R}^3 \to \mathbb{R}^3$$

$$L(\vec{u}) = r\vec{u} \text{ for } r > 1$$

4. contraction

$$L : \mathbb{R}^3 \to \mathbb{R}^3$$
$$L(\vec{u}) = r\vec{u} \text{ for } 0 < r < 1$$

5. counterclockwise rotation through an angle of ϕ

$$L : \mathbb{R}^2 \to \mathbb{R}^2$$
$$L(\vec{u}) = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \vec{u}$$

Exercise #2a Determine whether L is a linear transformation or not.

$$L : \mathbb{R}_3 \to \mathbb{R}_3$$
$$L([u_1 \quad u_2 \quad u_3]) = [u_1 \quad u_2^2 + u_3^2 \quad u_3^2]$$

Solution:

Let $\vec{u} = [u_1 \quad u_2 \quad u_3]$, $\vec{v} = [v_1 \quad v_2 \quad v_3] \in \mathbb{R}_3$

$$\begin{aligned} L(\vec{u} + \vec{v}) &= L([u_1 \quad u_2 \quad u_3] + [v_1 \quad v_2 \quad v_3]) \\ &= L([u_1 + v_1 \quad u_2 + v_2 \quad u_3 + v_3]) \\ &= \left[u_1 + v_1 \quad (u_2 + v_2)^2 + (u_3 + v_3)^2 \quad (u_3 + v_3)^2\right] \end{aligned}$$

$$\begin{aligned} L(\vec{u}) + L(\vec{v}) &= [u_1 \quad u_2^2 + u_3^2 \quad u_3^2] + [v_1 \quad v_2^2 + v_3^2 \quad v_3^2] \\ &= [u_1 + v_1 \quad u_2^2 + u_3^2 + v_2^2 + v_3^2 \quad u_3^2 + v_3^2] \end{aligned}$$

$$\Rightarrow L(\vec{u} + \vec{v}) \neq L(\vec{u}) + L(\vec{v})$$

Thus, L is NOT a linear transformation.

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W = set of all real-valued functions
V = subspace of all differentiable functions
$L : V \to W$ is defined by $L(f) = f'$. Show L is a linear transformation. (Exercise 24)

Proof:

(a) Let $f, g \in V$

$$L(f + g) = (f + g)' = f' + g' = L(f) + L(g)$$

(b) For $c \in \mathbb{R}$, $f \in V$

$$L(cf) = (cf)' = cf' = cL(f)$$

Thus, L is a linear transformation.

Theorem If $L: V \to W$ is a linear transformation, then

(a) $L\left(\vec{0}_V\right) = \vec{0}_W$

(b) $L\left(\vec{u} - \vec{v}\right) = L\left(\vec{u}\right) - L\left(\vec{v}\right)$, for $\vec{u}, \vec{v} \in V$

Proof: (watch the video)

We can describe what a linear transformation does for all the vectors in *V* by looking at what it does for a basis for *V*:

Theorem Let $L: V \to W$ be a linear transformation of an *n*-dimensional vector space *V* into a vector space *W*. Let $S = \left\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\right\}$ be a basis for *V*. If $\vec{v}$ is any vector in *V*, then $L\left(\vec{v}\right)$ is completely determined by $\left\{L\left(\vec{v}_1\right), L\left(\vec{v}_2\right), \ldots, L\left(\vec{v}_n\right)\right\}$.

Proof:

$\vec{v} \in V \Rightarrow \vec{v} = a_1\vec{v}_1 + a_2\vec{v}_2 + \ldots + a_n\vec{v}_n$, where a_i are real and unique

$\Rightarrow L\left(\vec{v}\right) = L\left(a_1\vec{v}_1 + a_2\vec{v}_2 + \ldots + a_n\vec{v}_n\right) = a_1L\left(\vec{v}_1\right) + a_2L\left(\vec{v}_2\right) + \ldots + a_nL\left(\vec{v}_n\right)$

$\Rightarrow L\left(\vec{v}\right)$ is completely determined by $\left\{L\left(\vec{v}_1\right), L\left(\vec{v}_2\right), \ldots, L\left(\vec{v}_n\right)\right\}$

Restatement of the Theorem

Let $L: V \to W$ and $L': V \to W$ be linear transformations of an *n*-dimensional vector space *V* into *W*. Let $S = \left\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\right\}$ be a basis for *V*.

If $L'\left(\vec{v}_i\right) = L\left(\vec{v}_i\right)$ for $i = 1, 2, \ldots, n$ then $L'\left(\vec{v}\right) = L\left(\vec{v}\right)$ for every $\vec{v} \in V$, i.e. if *L* and L' agree on a basis for *V* then *L* and L' are identical linear transformations.

Exercise #12 $L: \mathbb{R}_2 \to \mathbb{R}_2$, $L\left(\begin{bmatrix}1 & 1\end{bmatrix}\right) = \begin{bmatrix}1 & -2\end{bmatrix}$, $L\left(\begin{bmatrix}-1 & 1\end{bmatrix}\right) = \begin{bmatrix}2 & 3\end{bmatrix}$

Find $L\left(\begin{bmatrix}-1 & 5\end{bmatrix}\right)$.

Solution:

$$\begin{bmatrix}-1 & 5\end{bmatrix} = a\begin{bmatrix}1 & 1\end{bmatrix} + b\begin{bmatrix}-1 & 1\end{bmatrix}$$

$$\Rightarrow \begin{cases} a - b = -1 \\ a + b = 5 \end{cases} \Rightarrow a = 2, b = 3$$

$$\begin{bmatrix}-1 & 5\end{bmatrix} = 2\begin{bmatrix}1 & 1\end{bmatrix} + 3\begin{bmatrix}-1 & 1\end{bmatrix}$$

$$\begin{aligned} L\left(\begin{bmatrix}-1 & 5\end{bmatrix}\right) &= 2L\left(\begin{bmatrix}1 & 1\end{bmatrix}\right) + 3L\left(\begin{bmatrix}-1 & 1\end{bmatrix}\right) \\ &= 2\begin{bmatrix}1 & -2\end{bmatrix} + 3\begin{bmatrix}2 & 3\end{bmatrix} \\ &= \begin{bmatrix}2 & -4\end{bmatrix} + \begin{bmatrix}6 & 9\end{bmatrix} \\ &= \begin{bmatrix}8 & 5\end{bmatrix} \end{aligned}$$

Next, to show that if $L:\mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation then L must be a matrix transformation.

Theorem Let $L:\mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ be the natural basis for $\mathbb{R}^n$. Let A be the matrix whose jth column is $L(\vec{e}_j)$. If $\vec{x} = [x_1 \quad x_2 \quad \ldots \quad x_n]^T \in \mathbb{R}^n$, then

$$L(\vec{x}) = A\vec{x}. \qquad (1)$$

Moreover, A is the only matrix satisfying (1) and A is called the **standard matrix representing** L.

Proof:

$$\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + \ldots + x_n\vec{e}_n$$

$$L(\vec{x}) = x_1L(\vec{e}_1) + x_2L(\vec{e}_2) + \ldots + x_nL(\vec{e}_n)$$

$$L(\vec{x}) = A\vec{x}$$

Exercise 35: Show A is unique.

Exercise #6 Find A.

a) $L:\mathbb{R}^2 \to \mathbb{R}^2$

$$L\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = \begin{bmatrix} -u_2 \\ -u_1 \end{bmatrix}$$

Solution:

$$\left.\begin{matrix} L\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \\ L\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \end{matrix}\right\} \Rightarrow A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

Thus,

$$L\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ -1 \end{bmatrix} \text{ by definition}$$

or

$$L\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix} \text{ by using } A$$

b) $L:\mathbb{R}^2 \to \mathbb{R}^2$

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \to \begin{bmatrix} u_1 + ku_2 \\ u_2 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} 1 + k \cdot 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \to \begin{bmatrix} 0 + k \cdot 1 \\ 1 \end{bmatrix} = \begin{bmatrix} k \\ 1 \end{bmatrix}$$

Thus, $A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$.