**Math 270 Linear Algebra**

**Chapter 5 Inner Product Spaces**

**5.5 Orthogonal Complements**

**Chapter 4 Supplementary Exercises #34 and #35:**

$W_1, W_2$ subspaces of a vector space $V$

$\Rightarrow W_1 + W_2 = \left\{ \vec{v} \in W \middle| \vec{v} = \vec{w}_1 + \vec{w}_2, \vec{w}_1 \in W_1, \vec{w}_2 \in W_2 \right\}$ is a subspace of $V$

Moreover, if $V = W_1 + W_2$ and $W_1 \cap W_2 = \{\vec{0}\}$, then $V = W_1 \oplus W_2$, i.e. $V$ is the **<u>direct sum</u>** of $W_1$ and $W_2$ $\Rightarrow$ every vector in $V$ can be written uniquely as $\vec{w}_1 + \vec{w}_2$, $\vec{w}_1 \in W_1, \vec{w}_2 \in W_2$.

**Definition** Let $W$ be a subspace of an inner product space $V$. A vector $\vec{u} \in V$ is said to be **<u>orthogonal</u>** to $W$ if it is orthogonal to every vector in $W$. The set of all vectors in $V$ that are orthogonal to all vectors in $W$ is called the **<u>orthogonal complement</u>** of $W$ in $V$, denoted by $W^\perp$ ("$W$ perp").

$W^\perp = \left\{ \vec{v} \in V \middle| (\vec{v}, \vec{w}) = 0 \; \forall \vec{w} \in W \right\}$.

**Example** $W = span\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$ subspace of $\mathbb{R}^3 = \left\{ c \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, c \in \mathbb{R} \right\}$ = line

$$\vec{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \text{ is in } W^\perp \Leftrightarrow \vec{u} \perp c \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \text{ for any scalar } c$$

$$\Leftrightarrow \left( \vec{u}, c \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right) = 0$$

$$\Leftrightarrow \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \begin{bmatrix} c \\ 2c \\ 3c \end{bmatrix} \right) = 0$$

$$\Leftrightarrow cx + 2cy + 3cz = 0$$

$$\Leftrightarrow x + 2y + 3z = 0$$

$$\Leftrightarrow W^\perp \text{ is the plane with normal } \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\Leftrightarrow W^\perp = \left\{ (x, y, z) \middle| x + 2y + 3z = 0 \right\}$$

**Remarks**

1. $\vec{0}_V \in W^\perp$ (Exercise 24)
2. $V^\perp = \{\vec{0}_V\}$ (Exercise 25)
3. $\{\vec{0}_V\}^\perp = V$ (Exercise 25)

**Theorem** Let $W$ be a subspace of an inner product space $V$. Then

a) $W^{\perp}$ is a subspace of $V$.

b) $W \cap W^{\perp} = \{\vec{0}\}$

Proof:

a) Let $\vec{u}_1, \vec{u}_2 \in W^{\perp}$. Then $(\vec{u}_1, \vec{w}) = 0$, $(\vec{u}_2, \vec{w}) = 0 \;\; \forall \vec{w} \in W$

Consider $(\vec{u}_1 + \vec{u}_2, \vec{w}) = (\vec{u}_1, \vec{w}) + (\vec{u}_2, \vec{w}) = 0 + 0 = 0 \Rightarrow \vec{u}_1 + \vec{u}_2 \in W^{\perp}$

Next, let $\vec{u} \in W^{\perp}$ and $c$ be a real scalar .

$\vec{u} \in W^{\perp} \Rightarrow (\vec{u}, \vec{w}) = 0 \;\; \forall \vec{w} \in W$

Now, $(c\vec{u}, \vec{w}) = c(\vec{u}, \vec{w}) = c \cdot 0 = 0 \Rightarrow c\vec{u} \in W^{\perp}$.

Thus, $W^{\perp}$ is a subspace of $V$.

b) Let $\vec{u} \in W \cap W^{\perp}$. Then $\vec{u} \in W$ and $\vec{u} \in W^{\perp} \Rightarrow (\vec{u}, \vec{u}) = 0 \Rightarrow \vec{u} = \vec{0}$.

**Exercise #26** $W = spanS$, $\vec{u} \in W^{\perp} \Leftrightarrow \vec{u}$ is orthogonal to every vector in $S$.

**Example 2 p. 333** $V = P_3$

$$(p(t), q(t)) = \int_0^1 p(t)q(t)\,dt$$

$W$ = subspace of $P_3$ with basis $\{1, t^2\}$.

Find a basis for $W^{\perp}$.

Solution:

Let $p(t) = at^3 + bt^2 + ct + d \in W^{\perp}$

$$(p(t), 1) = \int_0^1 (at^3 + bt^2 + ct + d)\,dt = \frac{a}{4} + \frac{b}{3} + \frac{c}{2} + d = 0$$

Then $(p(t), t^2) = \int_0^1 (at^5 + bt^4 + ct^3 + dt^2)\,dt = \frac{a}{6} + \frac{b}{5} + \frac{c}{4} + \frac{d}{3} = 0$

$$\Rightarrow \begin{cases} 3a + 4b + 6c + 12d = 0 \\ 10a + 12b + 15c + 20d = 0 \end{cases}$$

$$\left[\begin{array}{cccc|c} 3 & 4 & 6 & 12 & 0 \\ 10 & 12 & 15 & 20 & 0 \end{array}\right] \xrightarrow{-R_1+R_2} \left[\begin{array}{cccc|c} 3 & 4 & 6 & 12 & 0 \\ 1 & 0 & -3 & -16 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cccc|c} 1 & 0 & -3 & -16 & 0 \\ 3 & 4 & 6 & 12 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & -3 & -16 & 0 \\ 0 & 4 & 15 & 60 & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & -3 & -16 & 0 \\ 0 & 1 & \frac{15}{4} & 15 & 0 \end{array}\right]$$

$$\Rightarrow \begin{cases} a = 3r + 16s \\ b = -\frac{15}{4}r - 15s \\ c = r \\ d = s \end{cases}$$

$$\Rightarrow p(t) = (3r+16s)t^3 + \left(-\tfrac{15}{4}r - 15s\right)t^2 + rt + s$$
$$= r\left(3t^3 - \tfrac{15}{4}t^2 + t\right) + s\left(16t^3 - 15t^2 + 1\right)$$
$$\Rightarrow \left\{3t^3 - \tfrac{15}{4}t^2 + t, 16t^3 - 15t^2 + 1\right\} \text{ spans } W^{\perp}$$
$$\Rightarrow \left\{3t^3 - \tfrac{15}{4}t^2 + t, 16t^3 - 15t^2 + 1\right\} \text{ is a basis for } W^{\perp} \text{ since the set is l.i.}$$

**Exercise #2** $W = span\left\{\begin{bmatrix}1\\2\\-1\end{bmatrix}, \begin{bmatrix}-1\\3\\2\end{bmatrix}\right\} = span\left\{\vec{v}_1, \vec{v}_2\right\}$. Find a basis for $W^{\perp}$.

Solution:

Let $\vec{w} = \begin{bmatrix}x\\y\\z\end{bmatrix} \in W^{\perp}$. Then

$\left(\vec{w}, \vec{v}_1\right) = 0 \Rightarrow x + 2y - z = 0$

$\left(\vec{w}, \vec{v}_2\right) = 0 \Rightarrow -x + 3y + 2z = 0$

$$\left[\begin{array}{ccc|c}1 & 2 & -1 & 0\\ -1 & 3 & 2 & 0\end{array}\right] \longrightarrow \left[\begin{array}{ccc|c}1 & 2 & -1 & 0\\ 0 & 5 & 1 & 0\end{array}\right] \longrightarrow \left[\begin{array}{ccc|c}1 & 0 & -\frac{7}{5} & 0\\ 0 & 1 & \frac{1}{5} & 0\end{array}\right]$$

$$\Rightarrow \begin{cases} x = \frac{7}{5}z \\ y = -\frac{1}{5}z \end{cases} \Rightarrow \begin{cases} x = \frac{7}{5}r \\ y = -\frac{1}{5}r \\ z = r, \ r \text{ real} \end{cases}$$

Basis for $W^{\perp} = \left\{\begin{bmatrix}\frac{7}{5}\\ -\frac{1}{5}\\ 1\end{bmatrix}\right\}$. $W^{\perp}$ is the set of all normals to the plane represented by $W$.

**Theorem** Let $W$ be a finite-dimensional subspace of an inner product space $V$. Then

$$V = W \oplus W^{\perp}.$$

Proof:

Let $\dim W = m$. Then $W$ has a basis consisting of $m$ vectors. By the G-S process, we can transform this basis to an ONB. Let $S = \left\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n\right\}$ be an ONB for $W$. If $\vec{v} \in V$, let

$$\vec{w} = \left(\vec{v}, \vec{w}_1\right)\vec{w}_1 + \left(\vec{v}, \vec{w}_2\right)\vec{w}_2 + \cdots + \left(\vec{v}, \vec{w}_m\right)\vec{w}_m \qquad (1)$$

and $$\vec{u} = \vec{v} - \vec{w} \qquad (2)$$

By (1), $\vec{w} \in W$. To show $\vec{u} \in W^{\perp}$:

$$\begin{aligned}(\vec{u},\vec{w}_i) &= (\vec{v}-\vec{w},\vec{w}_i)\\ &= (\vec{v},\vec{w}_i)-(\vec{w},\vec{w}_i)\\ &= (\vec{v},\vec{w}_i)-\left((\vec{v},\vec{w}_1)\vec{w}_1+(\vec{v},\vec{w}_2)\vec{w}_2+\cdots+(\vec{v},\vec{w}_m)\vec{w}_m,\vec{w}_i\right)\\ &= (\vec{v},\vec{w}_i)-(\vec{v},\vec{w}_i)(\vec{w}_i,\vec{w}_i)=0\end{aligned}$$

Since $(\vec{w}_i,\vec{w}_j)=0$ for $i \neq j \neq$ and $(\vec{w}_i,\vec{w}_i)=1,\ 1\leq 1\leq m$, then

$$\begin{aligned}\vec{u}\perp\vec{w}_i \text{ for all } i &\Rightarrow \vec{u}\perp \text{every vector in } W\\ &\Rightarrow \vec{u}\in W^{\perp}\\ &\Rightarrow \vec{v}=\vec{u}+\vec{w}\\ &\Rightarrow V=W+W^{\perp}\\ &\Rightarrow V=W\oplus W^{\perp}\end{aligned}$$

**Theorem** If $W$ is a finite-dimensional subspace of an inner product space $V$, then $(W^{\perp})^{\perp}=W$.

Proof:

(" $\supseteq$ ") $\vec{w}\in W \Rightarrow \vec{w}\perp\vec{u}\quad \forall\vec{u}\in W^{\perp}$

$\Rightarrow \vec{w}\in (W^{\perp})^{\perp}$

(" $\subseteq$ ") $\vec{v}\in (W^{\perp})^{\perp} \Rightarrow \vec{v}=\vec{w}+\vec{u},\ \ \vec{w}\in W,\ \vec{u}\in W^{\perp}$

$$\begin{aligned}\vec{u}\in W^{\perp} &\Rightarrow \vec{u}\perp\vec{v} \text{ and } \vec{u}\perp\vec{w}\\ &\Rightarrow 0=(\vec{u},\vec{v})=(\vec{u},\vec{w}+\vec{u})=(\vec{u},\vec{w})+(\vec{u},\vec{u})=(\vec{u},\vec{u})\\ &\Rightarrow (\vec{u},\vec{u})=0\\ &\Rightarrow \vec{u}=\vec{0}\\ &\Rightarrow \vec{v}=\vec{w}\\ &\Rightarrow \vec{v}\in W\end{aligned}$$

Hence, $(W^{\perp})^{\perp}=W$.

We say that $W$ and $W^{\perp}$ are **<u>orthogonal complements</u>**.

Given a matrix $A$, we can associate **4 fundamental vector spaces** with $A$:

Null space of $A$
Row space of $A$
Null space of $A^T$
Column space of $A$

**Theorem** If $A$ is any $m$ x $n$ matrix, then

a) null space $A = (\text{row space } A)^{\perp}$

b) null space $A^T = (\text{column space } A)^{\perp}$

Proof:

a) rank $A = r \Rightarrow$ dim null space $A = n - r$

dim row space $A = r \Rightarrow$ dim $(\text{row space } A)^{\perp} = n - r$

(i) $\vec{x} \in$ null space $A \Rightarrow A\vec{x} = \vec{0}$ .

Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m \in \mathbb{R}^n$ denote the rows of $A$

$\Rightarrow$ entries in the $m$ x 1 matrix $A\vec{x}$ are $\vec{v}_1\vec{x},\ \vec{v}_2\vec{x}, \ldots, \vec{v}_m\vec{x}$

$\Rightarrow \vec{v}_1\vec{x} = 0,\ \vec{v}_2\vec{x} = 0, \ldots, \vec{v}_m\vec{x} = 0$

$\Rightarrow \vec{x} \perp \underbrace{\vec{v}_1,\ \vec{v}_2, \ldots, \vec{v}_m}_{\text{these span the row space of } A}$

$\Rightarrow \vec{x} \in (\text{row space of } A)^{\perp}$

Thus, null space $A \subseteq (\text{row space of } A)^{\perp}$ .

(ii) $\vec{x} \in (\text{row space } A)^{\perp} \Rightarrow \vec{x} \perp \vec{v}_1,\ \vec{v}_2, \ldots, \vec{v}_m$

$\Rightarrow A\vec{x} = \vec{0}$

$\Rightarrow \vec{x} \in$ null space of $A$

Thus, $(\text{row space of } A)^{\perp} \subseteq$ null space $A$

(i) and (ii) $\Rightarrow (\text{row space of } A)^{\perp} =$ null space $A$

b) Replace $A$ by $A^T$ in (a).

**Exercise #10**

$$A = \begin{bmatrix} 2 & -1 & 3 & 4 \\ 0 & -3 & 7 & -2 \\ 1 & 1 & -2 & 3 \\ 1 & 4 & -9 & 5 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -3 & 7 & -2 \\ 2 & -1 & 3 & 4 \\ 1 & 4 & -9 & 5 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 1 & -2 & 3 \\ 0 & -3 & 7 & -2 \\ 0 & -3 & 7 & -2 \\ 0 & 3 & -7 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1/3 & 7/3 \\ 0 & 1 & -7/3 & 2/3 \\ 0 & 0 & 7 & -2 \\ 0 & 0 & -7 & 2 \end{bmatrix}$$

Solving $B\vec{x} = \vec{0}$ :

$$\left.\begin{array}{l} x_1 = -\frac{1}{3}r - \frac{7}{3}s \\ x_2 = \frac{7}{3}r - \frac{2}{3}s \\ x_3 = r \\ x_4 = s \end{array}\right\rangle \Rightarrow \text{basis for null space of } A = \left\{ \begin{bmatrix} -\frac{1}{3} \\ \frac{7}{3} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{7}{3} \\ -\frac{2}{3} \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\Rightarrow \text{basis for row space } A = \left\{ \begin{bmatrix} 1 & 0 & \frac{1}{3} & \frac{7}{3} \end{bmatrix}, \begin{bmatrix} 0 & 1 & -\frac{7}{3} & \frac{2}{3} \end{bmatrix} \right\}.$$

$$A^T = \begin{bmatrix} 2 & 0 & 1 & 1 \\ -1 & -3 & 1 & 4 \\ 3 & 7 & -2 & -9 \\ 4 & -2 & 3 & 5 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} -1 & -3 & 1 & 4 \\ 2 & 0 & 1 & 1 \\ 3 & 7 & -2 & -9 \\ 4 & -2 & 3 & 5 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 3 & -1 & -4 \\ 0 & -6 & 3 & 9 \\ 0 & -2 & 1 & 3 \\ 0 & -14 & 7 & 21 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1/2 & -3/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solving $C\vec{x} = \vec{0}$:

$$\left.\begin{aligned} x_1 &= -\tfrac{1}{2}r - \tfrac{1}{2}s \\ x_2 &= \tfrac{1}{2}r + \tfrac{3}{2}s \\ x_3 &= r \\ x_4 &= s \end{aligned}\right\rangle \Rightarrow \text{basis for null space of } A^T = \left\{ \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ \frac{3}{2} \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\Rightarrow \text{basis for column space } A = \left\{ \begin{bmatrix} 1 \\ 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -\frac{1}{2} \\ -\frac{3}{2} \end{bmatrix} \right\}.$$

**Example 4 p. 339**

$W = \text{span}\left\{\vec{w}_1, \vec{w}_2, \vec{w}_3, \vec{w}_4, \vec{w}_5\right\}$

where $\vec{w}_1 = [2 \quad -1 \quad 0 \quad 1 \quad 2]$, $\vec{w}_2 = [1 \quad 3 \quad 1 \quad -2 \quad -4]$, $\vec{w}_3 = [3 \quad 2 \quad 1 \quad -1 \quad -2]$

$\vec{w}_4 = [7 \quad 7 \quad 3 \quad -4 \quad -8]$, $\vec{w}_5 = [1 \quad -4 \quad -1 \quad -1 \quad -2]$

Find a basis for $W^\perp$.

Solution 1:

Let $\vec{u} = [a \quad b \quad c \quad d \quad e] \in W^\perp$

Solve

$$\left[\begin{array}{ccccc|c} 2 & -1 & 0 & 1 & 2 & 0 \\ 1 & 3 & 1 & -2 & -4 & 0 \\ 3 & 2 & 1 & -1 & -2 & 0 \\ 7 & 7 & 3 & -4 & -8 & 0 \\ 1 & -4 & -1 & -1 & -2 & 0 \end{array}\right] \Rightarrow S = \left\{ \begin{bmatrix} -\frac{1}{7} \\ -\frac{2}{7} \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\} \text{ basis for the null space of } A$$

$\Rightarrow$ basis for $W^\perp = \left\{\left[-\frac{1}{7} \quad -\frac{2}{7} \quad 1 \quad 0 \quad 0\right], [0 \quad 0 \quad 0 \quad -2 \quad 1]\right\}$

Solution 2:

Form

$$A = \begin{bmatrix} 2 & -1 & 0 & 1 & 2 \\ & & & & \\ & & & & \\ & & & & \end{bmatrix} \Rightarrow \text{row space } A = W \text{ and null space } A = W^\perp$$

To give a geometric interpretation for null space and row space:

**Example 5 p. 266**

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 2 & 1 & 3 \\ 7 & 1 & 8 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ row space of $A$ is a 2-dim subspace of $\mathbb{R}_3$ with basis $\{[1 \quad 0 \quad 1],[0 \quad 1 \quad 1]\}$

$a_1[1 \quad 0 \quad 1] + a_2[0 \quad 1 \quad 1] = [a_1 \quad a_2 \quad a_1 + a_2]$

$\Rightarrow$ row space of $A = \{(x, y, z) | z = x + y \text{ or } x + y - z = 0\}$ = plane thru $(0,0,0)$

Also, the null space of $A$ is a 1-dim sunbspace of $\mathbb{R}^3$ with basis $\left\{\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}\right\}$. This vector is orthogonal to the basis vectors for the row space of $A$. Thus,

$$(\text{row space of } A)^{\perp} = \text{null space of } A.$$

Next,

$$A^T = \begin{bmatrix} 3 & 2 & 7 \\ -1 & 1 & 1 \\ 2 & 3 & 8 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow \left\{\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}\right\}$ is a basis for the null space of $A^T$

$\Rightarrow$ null space of $A^T = \left\{ c \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \middle| c \text{ real} \right\}$

$\Rightarrow$ null space of $A^T$ is a line through the origin

Also,

$\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}\right\}$ is a basis for the column space of $A^T$

$\Rightarrow$ column space of $A^T$ is a plane thru the origin

Thus,

$$(\text{col space of } A^T)^{\perp} = \text{null space of } A^T.$$