Math 270 Linear Algebra

Chapter 5 Inner Product Spaces

5.4 Gram-Schmidt Process

Goal: To obtain an orthonormal basis for a Euclidean space *V*.

Theorem Let $S=\{\vec{u}_1,\vec{u}_2,\ldots,\vec{u}_n\}$ be an orthonormal basis for a Euclidean space *V* and let $\vec{v}\in V$. Then

$$\vec{v}=c_1\vec{u}_1+c_2\vec{u}_2+\cdots+c_n\vec{u}_n,$$

where $c_i=(\vec{v},\vec{u}_i)$, $i=1,2,\ldots,n$

Proof: Exercise 19

Example

$S=\left\{\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix},\begin{bmatrix}-\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix}\right\}=\{\vec{u}_1,\vec{u}_2\}$ is an orthonormal basis for $\mathbb{R}^2$.

Note: $(\vec{u}_1,\vec{u}_2)=0$ and $\|\vec{u}_1\|=1=\|\vec{u}_2\|$

Write $\vec{v}=\begin{bmatrix}3\\4\end{bmatrix}$ as a linear combination of $\vec{u}_1$ and $\vec{u}_2$.

Solution: According to the theorem,

$$\vec{v}=c_1\vec{u}_1+c_2\vec{u}_2,$$

where $c_1=(\vec{v},\vec{u}_1)$ and $c_2=(\vec{v},\vec{u}_2)$.

$$c_1=\left(\begin{bmatrix}3\\4\end{bmatrix},\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix}\right)=\frac{3}{\sqrt{2}}+\frac{4}{\sqrt{2}}=\frac{7}{\sqrt{2}}$$

$$c_2=\left(\begin{bmatrix}3\\4\end{bmatrix},\begin{bmatrix}-\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix}\right)=-\frac{3}{\sqrt{2}}+\frac{4}{\sqrt{2}}=\frac{1}{\sqrt{2}}$$

Thus,

$$\vec{v}=\frac{7}{\sqrt{2}}\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix}+\frac{1}{\sqrt{2}}\begin{bmatrix}-\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix}=\begin{bmatrix}\frac{7}{2}\\ \frac{7}{2}\end{bmatrix}+\begin{bmatrix}-\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}=\begin{bmatrix}3\\4\end{bmatrix}.$$

Hence, we don't need to solve a linear system to find c_1 and c_2 if we have an orthonormal basis.

Theorem (Gram-Schmidt Process)

Let V be an inner product space and $W \neq \{\vec{0}\}$ be an m-dimensional subspace of V. Then there exists an orthonormal basis $T = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m\}$ for W.

Proof: Watch the video.

Exercise #2 $\{\vec{u}_1, \vec{u}_2\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} \right\}$

$$\vec{v}_1 = \vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{(\vec{u}_2, \vec{v}_1)}{(\vec{v}_1, \vec{v}_1)} \vec{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{5}{2} \\ 1 \\ \frac{5}{2} \end{bmatrix} \quad \text{OR use} \begin{bmatrix} -5 \\ 2 \\ 5 \end{bmatrix}$$

$$S = \{\vec{v}_1, \vec{v}_2\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -\frac{5}{2} \\ 1 \\ \frac{5}{2} \end{bmatrix} \right\} \text{ is an orthogonal basis for } W \text{ or } \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 2 \\ 5 \end{bmatrix} \right\}$$

$$\|\vec{v}_1\| = \sqrt{1^2 + 0^2 + 1^2} = \sqrt{2}; \ \|\vec{v}_2\| = \sqrt{\tfrac{25}{4} + 1 + \tfrac{25}{4}} = \tfrac{\sqrt{54}}{2} \text{ or } \|\vec{v}_2\| = \sqrt{25 + 4 + 25} = \sqrt{54}$$

$$T = \{\vec{w}_1, \vec{w}_2\} = \left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} -\frac{5}{\sqrt{54}} \\ \frac{2}{\sqrt{54}} \\ \frac{5}{\sqrt{54}} \end{bmatrix} \right\} \text{ is an orthonormal basis}$$

Exercise #10 $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$

$$\vec{v}_1 = \vec{u}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{(\vec{u}_2,\vec{v}_1)}{(\vec{v}_1,\vec{v}_1)}\vec{v}_1 = \begin{bmatrix}0\\1\\1\end{bmatrix} - \frac{2}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}-\frac{2}{3}\\ \frac{1}{3}\\ \frac{1}{3}\end{bmatrix} \xrightarrow{\text{clear fractions}} \begin{bmatrix}-2\\1\\1\end{bmatrix}$$

$$\vec{v}_3 = \vec{u}_3 - \frac{(\vec{u}_3,\vec{v}_1)}{(\vec{v}_1,\vec{v}_1)}\vec{v}_1 - \frac{(\vec{u}_3,\vec{v}_2)}{(\vec{v}_2,\vec{v}_2)}\vec{v}_2 = \begin{bmatrix}1\\2\\3\end{bmatrix} - \frac{6}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} - \frac{3}{6}\begin{bmatrix}-2\\1\\1\end{bmatrix}$$

$$= \begin{bmatrix}1\\2\\3\end{bmatrix} - \begin{bmatrix}2\\2\\2\end{bmatrix} - \begin{bmatrix}-1\\ \frac{1}{2}\\ \frac{1}{2}\end{bmatrix} = \begin{bmatrix}0\\ -\frac{1}{2}\\ \frac{1}{2}\end{bmatrix} \xrightarrow{\text{clear fractions}} \begin{bmatrix}0\\-1\\1\end{bmatrix}$$

$$S = \{\vec{v}_1,\vec{v}_2,\vec{v}_3\} = \left\{\begin{bmatrix}1\\1\\1\end{bmatrix},\begin{bmatrix}-2\\1\\1\end{bmatrix},\begin{bmatrix}0\\-1\\1\end{bmatrix}\right\} \text{ is an orthogonal basis for } W$$

$$\|\vec{v}_1\| = \sqrt{3},\ \ \|\vec{v}_2\| = \sqrt{4+1+1} = \sqrt{6},\ \ \|\vec{v}_3\| = \sqrt{2}$$

$$T = \{\vec{w}_1,\vec{w}_2,\vec{w}_3\} = \left\{\begin{bmatrix}\frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{3}}\end{bmatrix},\begin{bmatrix}-\frac{2}{\sqrt{6}}\\ \frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{6}}\end{bmatrix},\begin{bmatrix}0\\ -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix}\right\} \text{ is an orthonormal basis for } W$$

Remark $(\vec{u},\vec{v}) = 0 \Rightarrow (\vec{u},c\vec{v}) = 0$ (Exercise 31)

V inner product space
$S = \{\vec{u}_1,\vec{u}_2,\dots,\vec{u}_n\}$ ONB

3 advantages of an ONB

1) $\vec{v} \in V \Rightarrow \vec{v} = a_1\vec{u}_1 + a_2\vec{u}_2 + \dots + a_n\vec{u}_n$ where $a_i = (\vec{v},\vec{u}_i)\ \ \forall i$

2)
$$\vec{v},\vec{w} \in V$$
$$\vec{v} = a_1\vec{u}_1 + a_2\vec{u}_2 + \cdots + a_n\vec{u}_n$$
$$\vec{w} = b_1\vec{u}_1 + b_2\vec{u}_2 + \cdots + b_n\vec{u}_n$$
$$\Rightarrow \underbrace{(\vec{v},\vec{w})}_{\substack{\text{arbitrary}\\ \text{inner product}}} = \underbrace{a_1b_1 + a_2b_2 + \cdots + a_nb_n}_{\text{standard inner product}}$$

3) $\left[\vec{v}\right]_S = \begin{bmatrix}a_1\\a_2\\ \vdots\\a_n\end{bmatrix} \Rightarrow \|\vec{v}\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$

Given any basis for an inner product space V, in order to find an ONB, use the Gram-Schmidt process.

Exercise #6 Let $S=\{t+1,t-1\}$ be a basis for a subspace *W* of P_2. Find an ONB for *W*.

Solution:

Use the inner product defined as follows:

$V=P,\ \ p(t),\ q(t)\in P$

$\left(p(t),q(t)\right)=\int_0^1 p(t)q(t)\,dt$

$S=\{t+1,t-1\}=\{\vec{u}_1,\vec{u}_2\}$

$\vec{v}_1=\vec{u}_1=t+1$

$$\vec{v}_2=\vec{u}_2-\frac{\left(\vec{u}_2,\vec{v}_1\right)}{\left(\vec{v}_1,\vec{v}_1\right)}\vec{v}_1$$

$$\left(\vec{u}_2,\vec{v}_1\right)=(t-1,t+1)=\int_0^1 (t-1)(t+1)\,dt$$

$$=\int_0^1\left(t^2-1\right)dt=\left[\frac{t^3}{3}-t\right]_0^1=\frac{1}{3}-1=-\frac{2}{3}$$

$$\left(\vec{v}_1,\vec{v}_1\right)=(t+1,t+1)=\int_0^1 (t+1)(t+1)\,dt$$

$$=\int_0^1 (t+1)^2\,dt=\left[\frac{(t+1)^3}{3}\right]_0^1=\frac{8}{3}-\frac{1}{3}=\frac{7}{3}$$

$$\vec{v}_2=t-1-\frac{-\frac{2}{3}}{\frac{7}{3}}(t+1)=t-1+\frac{2}{7}(t+1)=\frac{9}{7}t-\frac{5}{7}\xrightarrow{\text{clear fractions}}\vec{v}_2=9t-5$$

$$\left\|\vec{v}_1\right\|^2=\left(\vec{v}_1,\vec{v}_1\right)=\frac{7}{3}\Rightarrow\left\|\vec{v}_1\right\|=\sqrt{\frac{7}{3}}$$

$$\left\|\vec{v}_2\right\|^2=\left(\vec{v}_2,\vec{v}_2\right)=\int_0^1 (9t-5)^2\,dt=\frac{1}{9}\frac{(9t-5)^3}{3}\Bigg]_0^1=\frac{64}{27}+\frac{125}{27}=7\Rightarrow\left\|\vec{v}_2\right\|=\sqrt{7}$$

$S=\{t+1,9t-5\}$ is an orthogonal basis:

$$\int_0^1 (t+1)(9t-5)\,dt=\int_0^1\left(9t^2+4t-5\right)dt=0$$

$$T=\left\{\sqrt{\frac{3}{7}}(t+1),\frac{1}{\sqrt{7}}(9t-5)\right\}\text{ is an orthonormal basis.}$$

Remark If we change the order of the vectors in *S*, we might get a different ONB *T*.

Theorem Let *V* be an *n*-dimensional Euclidean space, and let $S=\{\vec{u}_1,\vec{u}_2,\ldots,\vec{u}_n\}$ be an ONB for *V*. If

$$\vec{v}=a_1\vec{u}_1+a_2\vec{u}_2+\cdots+a_n\vec{u}_n$$
$$\vec{w}=b_1\vec{u}_1+b_2\vec{u}_2+\cdots+b_n\vec{u}_n$$

then

$$\left(\vec{v},\vec{w}\right)=a_1b_1+a_2b_2+\cdots+a_nb_n$$

Note The theorem says that an arbitrary inner product in *V* behaves like the standard inner product in $\mathbb{R}^n$, when it is expressed in terms of coordinates with respect to an ONB.

Proof:

First compute

$C = [c_{ij}]$, where

$$c_{ij} = (\vec{u}_i, \vec{u}_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Hence, $C = I_n$. Also,

$$(\vec{v}, \vec{w}) = [\vec{v}]_S^T C [\vec{w}]_S = [\vec{v}]_S^T I_n [\vec{w}]_S = [\vec{v}]_S^T [\vec{w}]_S$$

$$= [a_1 \quad a_2 \quad \dots \quad a_n] \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$= a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

Remark If *T* is an ONB for an inner product space and $[\vec{v}]_T = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$, then $\|\vec{v}\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$.

Proof: Exercise 21

Exercise #22 Let *W* be the subspace of the Euclidean space $\mathbb{R}^3$ with basis $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} \right\}$.

Let $\vec{v} = \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} \in W$.

(a) Find $\|\vec{v}\|$ directly.

(b) Use G-S to find an ONB *T* for *W*.

(c) Find the length of $\vec{v}$ using $[\vec{v}]_T$.

Solution:

(a) $\|\vec{v}\| = \sqrt{(-1)^2 + 2^2 + (-3)^2} = \sqrt{14}$

(b)

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix},\ \vec{u}_2 = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$$

$$\vec{v}_1 = \vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\left(\vec{u}_2, \vec{v}_1\right)}{\left(\vec{v}_1, \vec{v}_1\right)} \vec{v}_1 = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} - \frac{-5}{5} \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}$$

$$\left\{\vec{v}_1, \vec{v}_2\right\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix} \right\} \text{ OB}$$

$$T = \left\{ \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \frac{1}{3} \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix} \right\} \text{ ONB}$$

(c)

$$T = \left\{ \begin{bmatrix} \frac{1}{\sqrt{5}} \\ 0 \\ -\frac{2}{\sqrt{5}} \end{bmatrix}, \begin{bmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{bmatrix} \right\} = \left\{\vec{w}_1, \vec{w}_2\right\}$$

$$\vec{v} = \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} = c_1 \vec{w}_1 + c_2 \vec{w}_2$$

where

$$c_1 = \left(\vec{v}, \vec{w}_1\right) = \left(\begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{5}} \\ 0 \\ -\frac{2}{\sqrt{5}} \end{bmatrix} \right) = -\frac{1}{\sqrt{5}} + \frac{6}{\sqrt{5}} = \frac{5}{\sqrt{5}} = \sqrt{5}$$

$$c_2 = \left(\vec{v}, \vec{w}_2\right) = \left(\begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{bmatrix} \right) = \frac{2}{3} + \frac{4}{3} + 1 = 3$$

$$\vec{v} = \sqrt{5}\vec{w}_1 + \vec{w}_2 \Rightarrow \left[\vec{v}\right]_T = \begin{bmatrix} \sqrt{5} \\ 3 \end{bmatrix}$$

$$\Rightarrow \left\|\vec{v}\right\| = \sqrt{\left(\sqrt{5}\right)^2 + 3^2} = \sqrt{14} \quad \text{same as (a)!}$$