Math 270 Linear Algebra

Chapter 5 Inner Product Spaces

5.3 Inner Product Spaces

Definition Let V be a real vector space. An **inner product** on V is a function that assigns to each ordered pair of vectors $\vec{u},\ \vec{v}\ \in V$ a real number $(\vec{u},\vec{v})$ satisfying

(a) $(\vec{u},\vec{u}) \geq 0$; $(\vec{u},\vec{u}) = 0 \Leftrightarrow \vec{u} = \vec{0}_V$

(b) $(\vec{v},\vec{u}) = (\vec{u},\vec{v})$ for any $\vec{u},\ \vec{v}\ \in V$

(c) $(\vec{u}+\vec{v},\vec{w}) = (\vec{u},\vec{w}) + (\vec{v},\vec{w})$ for any $\vec{u},\vec{v},\vec{w} \in V$

(d) $(c\vec{u},\vec{v}) = c(\vec{u},\vec{v})$ for $\vec{u},\vec{v} \in V$, c real scalar

Note

1) $(\vec{u},c\vec{v}) = (c\vec{v},\vec{u}) = c(\vec{v},\vec{u}) = c(\vec{u},\vec{v})$

2) $(\vec{u},\vec{v}+\vec{w}) = (\vec{u},\vec{v}) + (\vec{u},\vec{w})$

Example 1 $\mathbb{R}^n$

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \\ u_n \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \\ v_n \end{bmatrix}$$

$$(\vec{u},\vec{v}) = u_1v_1 + u_2v_2 + \ldots + u_nv_n$$

Remarks

1) Appendix B2: scalars could be complex numbers

2) $\vec{u},\ \vec{v}\ \in \mathbb{R}^n$, $(\vec{u},\vec{v}) = \vec{u}^T\vec{v}$

Example 2 V = finite-dimensional vector space

$S = \{\vec{u}_1,\ \vec{u}_2,\ldots,\vec{u}_n\}$ ordered basis for V

$\vec{v} = a_1\vec{u}_1 + a_2\vec{u}_2 + \ldots + a_n\vec{u}_n$

$\vec{w} = b_1\vec{u}_1 + b_2\vec{u}_2 + \ldots + b_n\vec{u}_n$

Define $(\vec{v},\vec{w}) = \left([\vec{v}]_S,[\vec{w}]_S\right) = a_1b_1 + a_2b_2 + \ldots + a_nb_n$. $(\vec{v},\vec{w})$ defines an inner product on V. (**Exercise 4)**

Need to show:

(a) $(\vec{v},\vec{v}) \geq 0$; $(\vec{v},\vec{v}) = 0 \Leftrightarrow \vec{v} = \vec{0}_V$

$$(\vec{v},\vec{v}) = \left([\vec{v}]_S,[\vec{v}]_S\right) = a_1a_1 + a_2a_2 + \cdots + a_na_n = a_1^2 + a_2^2 + \cdots + a_n^2 \geq 0$$

$$(\vec{v},\vec{v}) = 0 \Leftrightarrow a_1^2 + a_2^2 + \cdots + a_n^2 = 0 \Leftrightarrow a_1 = a_2 = \cdots = a_n = 0 \Leftrightarrow \vec{v} = \vec{0}_V$$

(b) $(\vec{v},\vec{w}) = (\vec{v},\vec{w})$ for any $\vec{v},\ \vec{w}\ \in V$

$$(\vec{v},\vec{w}) = \left([\vec{v}]_S, [\vec{w}]_S\right) = a_1b_1 + a_2b_2 + \ldots + a_nb_n$$
$$= b_1a_1 + b_2a_2 + \cdots + b_na_n = \left([\vec{w}]_S, [\vec{v}]_S\right) = (\vec{w},\vec{v})$$

(c) $(\vec{v}+\vec{w},\vec{u}) = (\vec{v},\vec{u}) + (\vec{w},\vec{u})$ for any $\vec{v},\vec{w},\vec{u} \in V$

Let $\vec{u} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$.

$$(\vec{v}+\vec{w},\vec{u}) = \left([\vec{v}+\vec{w}]_S, [\vec{u}]_S\right) = \left(\begin{bmatrix} a_1+b_1 \\ a_2+b_2 \\ \vdots \\ a_n+b_n \end{bmatrix}, \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}\right)$$
$$= (a_1+b_1)c_1 + (a_2+b_2)c_2 + \cdots + (a_n+b_n)c_n$$
$$= (a_1c_1 + a_2c_2 + \cdots + a_nc_n) + (b_1c_1 + b_2c_2 + \cdots + b_nc_n)$$
$$= \left([\vec{v}]_S, [\vec{u}]_S\right) + \left([\vec{w}]_S, [\vec{u}]_S\right)$$
$$= (\vec{v},\vec{u}) + (\vec{w},\vec{u})$$

(d) $(c\vec{v},\vec{w}) = c(\vec{v},\vec{w})$ for $\vec{v},\vec{w} \in V$, c real scalar

$$(c\vec{v},\vec{w}) = \left([c\vec{v}]_S, [\vec{w}]_S\right) = (ca_1)b_1 + (ca_2)b_2 + \cdots + (ca_n)b_n$$
$$= c(a_1b_1) + c(a_2b_2) + \cdots + c(a_nb_n)$$
$$= c(a_1b_1 + a_2b_2 + \cdots + a_nb_n)$$
$$= c\left([\vec{v}]_S, [\vec{w}]_S\right)$$
$$= c(\vec{v},\vec{w})$$

Example 3 Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},\ \ \vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\ \in \mathbb{R}^2$

Define $(\vec{u},\vec{v}) = u_1v_1 - u_2v_1 - u_1v_2 + 3u_2v_2$.

Show this gives an inner product on $\mathbb{R}^2$. (**Exercise 2)**

Solution:

(a) $(\vec{u},\vec{u}) = u_1^2 - 2u_1u_2 + 3u_2^2 = u_1^2 - 2u_1u_2 + u_2^2 + 2u_2^2 = (u_1 - u_2)^2 + 2u_2^2 \geq 0$

$(\vec{u},\vec{u}) = 0 \Rightarrow u_1 = u_2$ and $u_2 = 0$; $\ \vec{u} = \vec{0} \Rightarrow (\vec{u},\vec{u}) = 0$

(b) $\left(\vec{u},\vec{v}\right) = u_1v_1 - u_2v_1 - u_1v_2 + 3u_2v_2$

$\left(\vec{v},\vec{u}\right) = v_1u_1 - v_2u_1 - v_1u_2 + 3v_2u_2$

(c) $\left(\vec{u}+\vec{v},\vec{w}\right) = (u_1+v_1)w_1 - (u_2+v_2)w_1 - (u_1+v_1)w_2 + 3(u_2+v_2)w_2$

$= u_1w_1 - u_2w_1 - u_1w_2 + 3u_2w_2 + v_1w_1 - v_2w_1 - v_1w_2 + 3v_2w_2$

$= \left(\vec{u},\vec{w}\right) + \left(\vec{v},\vec{w}\right)$

(d) $\left(c\vec{u},\vec{v}\right) = cu_1v_1 - cu_2v_1 - cu_1v_2 + 3cu_2w_2$

$= c\left(u_1v_1 - u_2v_1 - u_1v_2 + 3u_2w_2\right) = c\left(\vec{u},\vec{v}\right)$

Example 4 $V = C[0,1]$ continuous real-valued functions on $[0,1]$.

For $f,g \in V$, let $(f,g) = \int_0^1 f(t)g(t)\,dt$.

Verify (a) – (d).

Solution:

(a) $(f,f) = \int_0^1 \left(f(t)\right)^2 dt \ge 0$

$(f,f) = 0 \Leftrightarrow f = 0$

(b) $(f,g) = \int_0^1 f(t)g(t)\,dt = \int_0^1 g(t)f(t)\,dt = (g,f)$

(c) $(f+g,h) = \int_0^1 \left[f(t)+g(t)\right]h(t)\,dt = \int_0^1 f(t)h(t)\,dt + \int_0^1 g(t)h(t)\,dt = (f,g)+(g,h)$

(d) $(cf,g) = \int_0^1 \left[cf(t)\right]g(t)\,dt = c\int_0^1 f(t)g(t)\,dt = c(f,g)$

Every inner product on a finite-dimensional vector space *V* is completely determined, in terms of a given basis, by a certain matrix.

Theorem Let $S = \left\{\vec{u}_1,\vec{u}_2,\dots,\vec{u}_n\right\}$ be an ordered basis for a finite-dimensional vector space *V* and assume that we are given an inner product on *V*. Let $c_{ij} = \left(\vec{u}_i,\vec{u}_j\right)$ and $C = \left[c_{ij}\right]$. Then

(a) *C* is a symmetric matrix.

(b) *C* determines $\left(\vec{v},\vec{w}\right)$ for every $\vec{v}$ and $\vec{w}$ in *V*.

Proof:

(a) $c_{ij} = \left(\vec{u}_i,\vec{u}_j\right) = \left(\vec{u}_j,\vec{u}_i\right) = c_{ji}$

(b) $\left(\vec{v},\vec{w}\right) \in V$

$\vec{v} = a_1\vec{u}_1 + a_2\vec{u}_2 + \dots + a_n\vec{u}_n;\ \vec{w} = b_1\vec{u}_1 + b_2\vec{u}_2 + \dots + b_n\vec{u}_n$

$$\Rightarrow \left[\vec{v}\right]_S = \begin{bmatrix} a_1 \\ a_2 \\ \\ a_n \end{bmatrix}, \quad \left[\vec{w}\right]_S = \begin{bmatrix} b_1 \\ b_2 \\ \\ b_n \end{bmatrix}$$

Then $\left(\vec{v},\vec{w}\right) = \left(\sum_{i=1}^{n} a_i\vec{u}_i,\vec{w}\right) = \sum_{i=1}^{n} a_i\left(\vec{u}_i,\vec{w}\right)$

$$= \sum_{i=1}^{n} a_i \left(\vec{u}_i, \sum_{j=1}^{n} b_j \vec{u}_j \right)$$
$$= \sum_{i=1}^{n} a_i \sum_{j=1}^{n} b_j \left(\vec{u}_i, \vec{u}_j \right)$$
$$= \sum_{i=1}^{n} \sum_{j=1}^{n} a_i b_j \left(\vec{u}_i, \vec{u}_j \right)$$
$$= \sum_{i=1}^{n} \sum_{j=1}^{n} a_i c_{ij} b_j$$
$$= \left[\vec{v} \right]_S^T C \left[\vec{w} \right]_S$$

For the standard inner product in $\mathbb{R}^n$:

If $A = \left[a_{ij} \right]$ is n x n and $\vec{x}, \vec{y} \in \mathbb{R}^n$ then $\left(A\vec{x}, \vec{y} \right) = \left(\vec{x}, A^T \vec{y} \right)$.

Proof:

$$\left(A\vec{x}, \vec{y} \right) = \left(A\vec{x} \right)^T \vec{y} = \left(\vec{x}^T A^T \right) \vec{y} = \vec{x}^T \left(A^T \vec{y} \right) = \left(\vec{x}, A^T \vec{y} \right)$$

Thus,

$$\left(\vec{v}, \vec{w} \right) = \left[\vec{v} \right]_S^T C \left[\vec{w} \right]_S = \left(\left[\vec{v} \right]_S, C \left[\vec{w} \right]_S \right).$$

Also,

$$\left(\vec{v}, \vec{w} \right) = \left(C \left[\vec{v} \right]_S, \left[\vec{w} \right]_S \right).$$

C is called the **matrix of the inner product with respect to the ordered basis *S***.

If $\vec{u} \in \mathbb{R}^n$, $\vec{u} \neq \vec{0}$, then $\left(\vec{u}, \vec{u} \right) > 0$ and letting $\vec{x} = \left[\vec{u} \right]_S$ we get

$$\vec{x}^T C \vec{x} > 0 \quad \forall \text{ nonzero vector } \vec{x} \in \mathbb{R}^n .$$

An n x n symmetric matrix *C* with this property is called **positive definite**.

Note: *C* is nonsingular.

Proof: *C* singular $\Rightarrow \exists\ \vec{x}_0 \neq \vec{0}$ s.t. $A\vec{x}_0 = \vec{0} \Rightarrow \vec{x}_0^T A \vec{x}_0 = \vec{x}_0^T \vec{0} = 0$, a contradiction.

If $C = \left[c_{ij} \right]$ is an n x n positive definite matrix, then we can use *C* to define an inner product on *V*:

$$\left(\vec{v}, \vec{w} \right) = \left(\left[\vec{v} \right]_S, C \left[\vec{w} \right]_S \right) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i c_{ij} b_j .$$

Exercise #24 Consider the inner product defined in Example 3 p. 308

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^2$$

Define $(\vec{u}, \vec{v}) = u_1v_1 - u_2v_1 - u_1v_2 + 3u_2v_2$. Choose an ordered basis *S* for $\mathbb{R}^2$ and find the matrix of the inner product with respect to *S*.

Solution:

Let $S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} = \{\vec{e}_1, \vec{e}_2\}$

$$C = [c_{ij}], \quad c_{ij} = (\vec{e}_i, \vec{e}_j)$$

$$c_{11} = (\vec{e}_1, \vec{e}_1) = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = (1)(1) - (0)(1) - (1)(0) + 3(0)(0) = 1$$

$$c_{12} = (\vec{e}_1, \vec{e}_2) = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = (1)(0) - (0)(0) - (1)(1) + 3(0)(1) = -1$$

$$c_{21} = (\vec{e}_2, \vec{e}_1) = \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = (0)(1) - (1)(1) - (0)(0) + 3(1)(0) = -1$$

$$c_{22} = (\vec{e}_2, \vec{e}_2) = \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = (0)(0) - (0)(1) - (1)(0) + 3(1)(1) = 3$$

$$C = \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}$$

C is symmetric. To check if *C* is positive-definite:

For $\vec{0} \neq \vec{x} \in \mathbb{R}^2$, show that $\vec{x}^T C \vec{x} > 0$:

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \boxed{x_1 - x_2} & \boxed{-x_1 + 3x_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= x_1^2 - x_1x_2 - x_1x_2 + 3x_2^2$$

$$= (x_1^2 - 2x_1x_2 + x_2^2) + 2x_2^2$$

$$= (x_1 - x_2)^2 + 2x_2^2 > 0$$

So, for example, if $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$$(\vec{u}, \vec{v}) = (1)(2) - (2)(2) - (1)(3) + 3(2)(3) = 2 - 4 - 3 + 18 = 20 - 7 = 13,$$

or, using *C*:

$$\left(\vec{u},\vec{v}\right)=\left(\left[\vec{u}\right]_S,C\left[\vec{v}\right]_S\right)=\left(\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}1&-1\\-1&3\end{bmatrix}\begin{bmatrix}2\\3\end{bmatrix}\right)=\left(\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}-1\\7\end{bmatrix}\right)$$
$$=(1)(-1)+(2)(7)=-1+14=13 \text{ SAME!}$$

Example 7 p. 311

Let $C=\begin{bmatrix}2&1\\1&2\end{bmatrix}$

$$\vec{x}^T C\vec{x}=\begin{bmatrix}x_1 & x_2\end{bmatrix}\begin{bmatrix}2&1\\1&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$$
$$=2x_1^2+2x_1x_2+2x_2^2$$
$$=x_1^2+x_2^2+\left(x_1+x_2\right)^2>0 \text{ if } \vec{x}\neq\vec{0}$$

Define an inner product on P_1 whose matrix with respect to the ordered basis $S=\{t,1\}$ is C.
Let $p(t)=a_1t+a_2$, $q(t)=b_1t+b_2$. Let $\left(p(t),q(t)\right)=2a_1b_1+a_2b_1+a_1b_2+2a_2b_2$.

Verify the properties:

(a) $\left(p(t),p(t)\right)=2a_1^2+2a_1a_2+2a_2^2=a_1^2+a_2^2+\left(a_1+a_2\right)^2\geq 0$

$\left(p(t),p(t)\right)=0\Leftrightarrow a_1=a_2=0\Leftrightarrow p(t)=0$

(b) $\left(p(t),q(t)\right)=2a_1b_1+a_2b_1+a_1b_2+2a_2b_2$

$\left(q(t),p(t)\right)=2b_1a_1+b_2a_1+b_1a_2+2b_2a_2$

(c) $\left(p(t)+q(t),h(t)\right)=\left[\left(a_1+b_1\right)t+a_2+b_2\right]\cdot\left[c_1t+c_2\right]$

$$=2\left(a_1+b_1\right)c_1+\left(a_2+b_2\right)c_1+\left(a_1+b_1\right)c_2+2\left(a_2+b_2\right)c_2$$
$$=2a_1c_1+2b_1c_1+a_2c_1+b_2c_1+a_1c_2+b_1c_2+2a_2c_2+2b_2c_2$$
$$=\left(p(t),h(t)\right)+\left(g(t),h(t)\right)$$

(d) $\left(cp(t),q(t)\right)=2ca_1b_1+ca_2b_1+ca_1b_2+2ca_2b_2$

$$=c\left(2a_1b_1+a_2b_1+a_1b_2+2a_2b_2\right)$$
$$=c\left(p(t),q(t)\right)$$

Definition A real vector space that has an inner product defined on it is an **inner product space**. If the space is finite-dimensional, it is called a **Euclidean space**.

Theorem (Cauchy-Schwarz Inequality)

$\vec{u},\vec{v}\in V$, V an inner product space

$$\left|\left(\vec{u},\vec{v}\right)\right|\leq\left\|\vec{u}\right\|\left\|\vec{v}\right\|$$

Proof:

(i) $\vec{u}=\vec{0}\Rightarrow\left\|\vec{u}\right\|=0\Rightarrow\left(\vec{u},\vec{v}\right)=0$ by **Exercise 7b**

(ii) $\vec{u}\neq\vec{0}$:

Consider $r\vec{u}+\vec{v}$, r scalar

$$0 \le \left(r\vec{u}+\vec{v}, r\vec{u}+\vec{v}\right) = \left(\vec{u},\vec{u}\right)r^2 + 2r\left(\vec{u},\vec{v}\right) + \left(\vec{v},\vec{v}\right)$$
$$= ar^2 + 2br + c$$

where $a=\left(\vec{u},\vec{u}\right)$, $b=\left(\vec{u},\vec{v}\right)$, $c=\left(\vec{v},\vec{v}\right)$

If we fix $\vec{u}$ and $\vec{v}$,

$$ar^2 + 2br + c = p(r)$$

is a quadratic polynomial in r that has nonnegative values for all values of r. Then $p(r)$ has at most one real root. The roots of $p(r)$ are given by

$$\frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \frac{-b-\sqrt{b^2-4ac}}{2a}, \ \left(a \ne 0 \text{ since } \vec{u} \ne 0\right).$$

Then,

$$b^2 - ac \le 0 \Rightarrow b^2 \le ac$$
$$\Rightarrow |b| \le \sqrt{a}\sqrt{c}$$
$$\Rightarrow \left|\left(\vec{u},\vec{v}\right)\right| \le \sqrt{\left(\vec{u},\vec{u}\right)}\sqrt{\left(\vec{v},\vec{v}\right)}$$
$$\Rightarrow \left|\left(\vec{u},\vec{v}\right)\right| \le \|\vec{u}\|\|\vec{v}\|$$

Remark If $\vec{u} \ne \vec{0}$, $\vec{v} \ne \vec{0}$ in an inner product space V, the Cauchy-Schwarz inequality can be written as

$$-1 \le \frac{\left(\vec{u},\vec{v}\right)}{\|\vec{u}\|\|\vec{v}\|} \le 1.$$

This implies that there is one and only one angle θ such that

$$\cos\theta = \frac{\left(\vec{u},\vec{v}\right)}{\|\vec{u}\|\|\vec{v}\|}, \quad 0 \le \theta \le \pi .$$

θ is called the **angle between** $\vec{u}$ and $\vec{v}$.

Triangle Inequality $\|\vec{u}+\vec{v}\| \le \|\vec{u}\| + \|\vec{v}\|$

Proof:

$$
\begin{aligned}
\|\vec{u}+\vec{v}\|^2 &= \left(\vec{u}+\vec{v},\vec{u}+\vec{v}\right) \\
&= \left(\vec{u},\vec{u}\right)+2\left(\vec{u},\vec{v}\right)+\left(\vec{v},\vec{v}\right) \\
&= \|\vec{u}\|^2+2\left(\vec{u},\vec{v}\right)+\|\vec{v}\|^2 \\
&\le \|\vec{u}\|^2+2\left|\left(\vec{u},\vec{v}\right)\right|+\|\vec{v}\|^2 \\
&\le \|\vec{u}\|^2+2\|\vec{u}\|\|\vec{v}\|+\|\vec{v}\|^2 \\
&= \left(\|\vec{u}\|+\|\vec{v}\|\right)^2
\end{aligned}
$$

$$\Rightarrow \|\vec{u}+\vec{v}\| \le \|\vec{u}\|+\|\vec{v}\|$$

Definition The **distance** between two vectors $\vec{u}$ and $\vec{v} \in V$ is defined as $d\left(\vec{u},\vec{v}\right)=\|\vec{u}-\vec{v}\|$.

Definition $\vec{u}$ and $\vec{v} \in V$ are **orthogonal** if $\left(\vec{u},\vec{v}\right)=0$.

Example $\begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}0\\-1\\0\end{bmatrix} \in \mathbb{R}^3$ are orthogonal

Remarks

1. $\vec{0}_V$ is orthogonal to every vector in V.
2. $\vec{u}$ and $\vec{v}$ are orthogonal $\Leftrightarrow$ the angle between them is $\dfrac{\pi}{2}$.
3. The set of vectors in V that are orthogonal to a fixed vector in V is a subspace of V. (Exercise 23)

Definition Let V be an inner product space. A set S of vectors in V is called **orthogonal** if any two distinct vectors in S are orthogonal. If, in addition, each vector in S is of unit length, then S is called **orthonormal**.

$\vec{x} \ne \vec{0}$, $\vec{u}=\dfrac{1}{\|\vec{x}\|}\vec{x}$ unit vector in the same direction as $\vec{x}$

$$\|\vec{u}\|=\sqrt{\left(\vec{u},\vec{u}\right)}=\sqrt{\left(\frac{\vec{x}}{\|\vec{x}\|},\frac{\vec{x}}{\|\vec{x}\|}\right)}=\sqrt{\frac{\left(\vec{x},\vec{x}\right)}{\|\vec{x}\|\|\vec{x}\|}}=\sqrt{\frac{\|\vec{x}\|^2}{\|\vec{x}\|\|\vec{x}\|}}=1$$

$\cos\theta=1 \Rightarrow \vec{x}$ and $\vec{u}$ have the same direction

Theorem Let $S=\left\{\vec{u}_1,\vec{u}_2,\ldots,\vec{u}_n\right\}$ be a finite orthogonal set of nonzero vectors in an inner product space V. Then S is linearly independent.

Proof:

Let $a_1\vec{u}_1+a_2\vec{u}_2+\ldots+a_n\vec{u}_n=\vec{0}$.

Take the inner product of both sides with $\vec{u}_i$:

$$\left(a_1\vec{u}_1 + a_2\vec{u}_2 + \ldots + a_n\vec{u}_n, \vec{u}_i\right) = \left(\vec{0}, \vec{u}_i\right)$$

$$\Rightarrow a_1\left(\vec{u}_1, \vec{u}_i\right) + a_2\left(\vec{u}_2, \vec{u}_i\right) + \ldots a_i\left(\vec{u}_i, \vec{u}_i\right) + \ldots . + a_n\left(\vec{u}_n, \vec{u}_i\right) = 0$$

$$\Rightarrow a_i\left(\vec{u}_i, \vec{u}_i\right) = 0$$

$\Rightarrow a_i = 0$ since $\vec{u}_i \neq \vec{0}$

Repeating this for $i = 1,2,\ldots,n$, we get

$$a_1 = a_2 = \ldots = a_n = 0$$

and thus S is linearly independent.