Math 270 Linear Algebra

Chapter 5 Inner Product Spaces

5.1 Length and Direction in $\mathbb{R}^2$ and $\mathbb{R}^3$

Length

$\mathbb{R}^2$ $\quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \qquad \| \vec{v} \| = \sqrt{v_1^2 + v_2^2}$ **length** or **magnitude**

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$\|\vec{v} - \vec{u}\| = \sqrt{(v_1 - u_1)^2 + (v_2 - u_2)^2}$ **distance between** $\vec{u}$ and $\vec{v}$

Note: $\vec{v} - \vec{u} = \begin{bmatrix} v_1 - u_1 \\ v_2 - u_2 \end{bmatrix}$

$\mathbb{R}^3$ $\quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \quad \| \vec{v} \| = \sqrt{v_1^2 + v_2^2 + v_3^2}$ **length** of $\vec{v}$

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$\|\vec{v} - \vec{u}\| = \sqrt{(v_1 - u_1)^2 + (v_2 - u_2)^2 + (v_3 - u_3)^2}$ **distance between** $\vec{u}$ and $\vec{v}$

Direction

$\mathbb{R}^2$

$\mathbb{R}^3$

Given $\vec{u}, \ \vec{v} \in \mathbb{R}^2 \left(\mathbb{R}^3\right)$, find the angle $\theta, \ 0 \le \theta \le \pi,$ between $\vec{u}$ and $\vec{v}$.

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

By the law of cosines,

$$\left\| \vec{v} - \vec{u} \right\|^2 = \left\| \vec{u} \right\|^2 + \left\| \vec{v} \right\|^2 - 2\left\| \vec{u} \right\| \left\| \vec{v} \right\| \cos\theta$$

$$\Rightarrow \cos\theta = \frac{\left\| \vec{u} \right\|^2 + \left\| \vec{v} \right\|^2 - \left\| \vec{v} - \vec{u} \right\|^2}{2\left\| \vec{u} \right\| \left\| \vec{v} \right\|}$$

$$= \frac{u_1^2 + u_2^2 + u_3^2 + v_1^2 + v_2^2 + v_3^2}{2\left\| \vec{u} \right\| \left\| \vec{v} \right\|} - \frac{(v_1 - u_1)^2 + (v_2 - u_2)^2 + (v_3 - u_3)^2}{2\left\| \vec{u} \right\| \left\| \vec{v} \right\|}$$

$$\Rightarrow \cos\theta = \frac{u_1v_1 + u_2v_2 + u_3v_3}{\left\| \vec{u} \right\| \left\| \vec{v} \right\|}$$

Similarly, for $\vec{u}, \vec{v} \in \mathbb{R}^2$,

$$\cos\theta = \frac{u_1v_1 + u_2v_2}{\left\| \vec{u} \right\| \left\| \vec{v} \right\|}, \quad 0 \le \theta \le \pi .$$

Note $\vec{0}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ has no direction.

<u>Standard inner product</u> or **<u>dot product</u>** on $\mathbb{R}^2$ or $\mathbb{R}^3$

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^2 \Rightarrow \vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2$$

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbb{R}^3 \Rightarrow \vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3$$

Note:

1) $\left\| \vec{v} \right\|^2 = \vec{v} \cdot \vec{v}$

2) $\cos\theta = \dfrac{\vec{u} \cdot \vec{v}}{\left\| \vec{u} \right\| \left\| \vec{v} \right\|}, \quad 0 \le \theta \le \pi$

3) $-1 \le \dfrac{\vec{u} \cdot \vec{v}}{\left\| \vec{u} \right\| \left\| \vec{v} \right\|} \le 1$

4) $\vec{u}$ and $\vec{v}$ are **<u>orthogonal</u>** or **<u>perpendicular</u>** $\Leftrightarrow \vec{u} \cdot \vec{v} = 0$.

Properties of the standard inner product

Theorem Let $\vec{u}, \vec{v}$ be vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$, *c* scalar. Then

(a) $\vec{u}\cdot\vec{v} \geq 0;\ \vec{u}\cdot\vec{u} = 0 \Leftrightarrow \vec{u} = \vec{0}$
(b) $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$
(c) $\left(\vec{u}+\vec{v}\right)\cdot\vec{w} = \vec{u}\cdot\vec{w}+\vec{v}\cdot\vec{w}$
(d) $c\vec{u}\cdot\vec{v} = c\left(\vec{u}\cdot\vec{v}\right)$

Unit Vectors

A **<u>unit vector</u>** in $\mathbb{R}^2$ or $\mathbb{R}^3$ is a vector whose length is 1.

If $\vec{x} \neq \vec{0}$, the vector $\vec{u} = \frac{1}{\left\|\vec{x}\right\|}\vec{x} = \frac{\vec{x}}{\left\|\vec{x}\right\|}$ is the **<u>unit vector in the direction of</u>** $\left\|\vec{x}\right\|$.

$\mathbb{R}^2$: 2 special unit vectors $\vec{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\mathbb{R}^3$: 3 special unit vectors $\vec{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$