Math 270 Linear Algebra

Chapter 4 Real Vector Spaces

4.9 Rank of a Matrix

Definition
Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \text{ be } m \times n.$$

The rows of A, considered as vectors in $\mathbb{R}_n$, span a subspace of $\mathbb{R}_n$, called the **row space** of A. Similarly, the columns of A, considered as vectors in $\mathbb{R}^m$, span a subspace of $\mathbb{R}^m$, called the **column space** of A.

Theorem A and B row (column) equivalent $\Rightarrow$ row (col) space of A = row (col) space of B

The theorem can be used to find a basis for a subspace spanned by a given set of vectors:

Example 1

$$A = \begin{bmatrix} 1 & 1 & -1 & 2 & 0 \\ 2 & -4 & 0 & 1 & 1 \\ 5 & -1 & -3 & 7 & 1 \\ 3 & -9 & 1 & 0 & 2 \end{bmatrix} \begin{matrix} \vec{v}_1 \\ \vec{v}_2 \\ \vec{v}_3 \\ \vec{v}_4 \end{matrix}$$

Find a basis for the subspace of $\mathbb{R}_5$ spanned by $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$.

Solution:

$$\begin{bmatrix} 1 & 1 & -1 & 2 & 0 \\ 2 & -4 & 0 & 1 & 1 \\ 5 & -1 & -3 & 7 & 1 \\ 3 & -9 & 1 & 0 & 2 \end{bmatrix} \xrightarrow{\text{pivot on } a_{11}} \begin{bmatrix} 1 & 1 & -1 & 2 & 0 \\ 0 & -6 & 2 & -3 & 1 \\ 0 & -6 & 2 & -3 & 1 \\ 0 & -12 & 4 & -6 & 2 \end{bmatrix}$$

$$\xrightarrow[-2R_2+R_4]{-R_2+R_3} \begin{bmatrix} 1 & 1 & -1 & 2 & 0 \\ 0 & -6 & 2 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, a basis for V is $\{[1 \;\; 1 \;\; -1 \;\; 2 \;\; 0], [0 \;\; -6 \;\; 2 \;\; -3 \;\; 1]\}$.
Note, however, that these vectors are not in the original spanning set.

Ex #2 Find a basis for the subspace of P_3 spanned by

$$S = \{t^3 + t^2 + 2t + 1,\ t^3 - 3t + 1,\ t^2 + t + 2,\ t + 1,\ t^3 + 1\}.$$

Solution:

$P_3 \cong \mathbb{R}_4$ under $L(at^3 + bt^2 + ct + d) = [a \quad b \quad c \quad d] \Rightarrow L(V) \cong W \subseteq \mathbb{R}_4$

W is spanned by $\{L(\vec{v}_1), L(\vec{v}_2), L(\vec{v}_3), L(\vec{v}_4), L(\vec{v}_5)\}$

$\Rightarrow W$ is the row space of

$$A = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 0 & -3 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{pivot on } a_{11}} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & -5 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & -1 & -2 & 0 \end{bmatrix} \xrightarrow{\text{pivot on } a_{22}} \begin{bmatrix} 1 & 0 & -3 & 1 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & -4 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

$$\xrightarrow{\text{pivot on } a_{43}} \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -3 \end{bmatrix} \xrightarrow{\text{pivot on } a_{54}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} t^3 \\ t^2 \\ \\ t \\ 1 \end{matrix}$$

Thus, a basis is $\{t^3, t^2, t, 1\}$.

Definition The dimension of the row (col) space of A is called the **row (col) rank** of A.

Note:

1. A row (col) equivalent to $B \Rightarrow$ row (col) rank A = row (col) rank B
2. B in r.r.e.f $\Rightarrow$ row rank B = number of nonzero rows

So, to find the row rank of A…

Ex # 10a

$$\begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 0 & 5 & 4 & 0 & -1 \\ 2 & -1 & 2 & 4 & 3 \end{bmatrix} \xrightarrow{\text{pivot on } a_{11}} \begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 0 & 5 & 4 & 0 & -1 \\ 0 & -5 & -4 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 + R_3} \begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 0 & 5 & 4 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{may stop here} \Rightarrow \textit{row rank } A = 2$$

To find the column rank of A, get the row rank of A^T:

$$\begin{bmatrix} 1 & 0 & 2 \\ 2 & 5 & -1 \\ 3 & 4 & 2 \\ 2 & 0 & 4 \\ 1 & -1 & 3 \end{bmatrix} \xrightarrow{\text{pivot on } a_{11}} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 5 & -5 \\ 0 & 4 & -4 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow{\text{pivot on } a_{22}} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{row rank } A^T = 2 = \text{col rank } A$$

Note: row rank A = col rank A. This is true in general:

Theorem A m x n; row rank A = col rank A
Proof: Read.

Definition **<u>rank</u>** A = row (col) rank A

Theorem A equivalent to $B \Leftrightarrow$ rank A = rank B

Proof:
$("\Rightarrow")$ Recall from Chapter 2:
A equivalent to $B \Leftrightarrow \exists$ nonsingular matrices P and Q s.t. $B = PAQ$.
Thus,
A equivalent to $B \Rightarrow rank\ B = rank(PAQ) = rank(PA) = rank\ A$

$("\Leftarrow")$ rank A = rank B = r

$\Rightarrow \exists$ nonsingular matrices P_1, Q_1, P_2, Q_2 s.t. $P_1AQ_1 = \begin{bmatrix} I_r & O \\ O & O \end{bmatrix} = P_2BQ_2$

$\Rightarrow P_1AQ_1 = P_2BQ_2$

$\Rightarrow B = \underbrace{P_2^{-1}P_1}_{P} A \underbrace{Q_1Q_2^{-1}}_{Q}$, P and Q both nonsingular

$\Rightarrow B$ is equivalent to A

Theorem If A is m x n, then rank A + nullity A = n.

Example Back to the previous example,

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 0 & 5 & 4 & 0 & -1 \\ 2 & -1 & 2 & 4 & 3 \end{bmatrix};\ \text{rank } A = 2$$

$$A\vec{x} = \vec{0} : \left[\begin{array}{ccccc|c} 1 & 2 & 3 & 2 & 1 & 0 \\ 0 & 5 & 4 & 0 & -1 & 0 \\ 2 & -1 & 2 & 4 & 3 & 0 \end{array}\right] \ldots \longrightarrow \left[\begin{array}{ccccc|c} 1 & 2 & 3 & 2 & 1 & 0 \\ 0 & 5 & 4 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\text{pivot on } a_{22}} \left[\begin{array}{ccccc|c} 1 & 0 & \frac{7}{5} & 2 & \frac{3}{5} & 0 \\ 0 & 1 & \frac{4}{5} & 0 & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned} x_1 &= -\tfrac{7}{5}r - 2s - \tfrac{3}{5}t \\ x_2 &= -\tfrac{4}{5}r \qquad - \tfrac{1}{5}t \\ \Rightarrow x_3 &= r \\ x_4 &= \qquad s \\ x_5 &= \qquad\qquad t\partial \end{aligned} \quad \Rightarrow \text{solutions look like} \begin{bmatrix} -\frac{7}{5}r - 2s - \frac{3}{5}t \\ -\frac{4}{5}r \qquad - \frac{1}{5}t \\ r \\ s \\ t \end{bmatrix} \Rightarrow \text{dim sol space } = 3 \text{ or nullity } = 3$$

$2 + 3 = 5$, the number of columns of A

Rank and Singularity

Theorem If A is n x n, then rank $A = n \Leftrightarrow A$ is row equivalent to I_n.

Proof:

$("\Rightarrow")$ rank $A = n$

$\Rightarrow A$ is row equivalent to B in r.r.e.f.E and rank $B = n$

$\Rightarrow B$ has no zero rows

$\Rightarrow B = I_n$

$\Rightarrow A$ is row equivalent to I_n

$("\Leftarrow")$ A row equivalent to $I_n \Rightarrow$ rank $A = rank\ I_n = n$

Corollary A is nonsingular $\Leftrightarrow$ rank $A = n$.

Corollary $Ax = \vec{0}$, A $n\,x\,n$, has a nontrivial solution $\Leftrightarrow$ rank $A < n$.

Corollary A n x n, $A\vec{x} = \vec{b}$ has a unique solution for every n x 1 matrix $\vec{b} \Leftrightarrow$ rank $A = n$.

Remarks:

1. Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a set of vectors in $\mathbb{R}_n$

   A is the matrix whose jth row is $\vec{v}_j$

   Then S is linearly independent iff rank $A = n$. (Exercise #)

2. Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a set of vectors in $\mathbb{R}_n$

   A is the matrix whose jth column is $\vec{v}_j$

   Then S is linearly independent iff rank $A = n$. (Exercise #).

Applications of Rank to $A\vec{x} = \vec{b}$:

Theorem $A\vec{x} = \vec{b}$ has a solution $\Leftrightarrow$ $\mathit{rank}\ A = \mathit{rank}\ \left[A \mid \vec{b}\right]$.

Example

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & -2 & 2 \\ 0 & 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 1 & -2 & 2 & 2 \\ 0 & 1 & 3 & 3 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & -2 & 2 & 2 \\ 2 & 1 & 3 & 1 \\ 0 & 1 & 3 & 3 \end{array}\right] \xrightarrow{\text{pivot on } a_{11}} \left[\begin{array}{ccc|c} 1 & -2 & 2 & 2 \\ 0 & 5 & -1 & -3 \\ 0 & 1 & 3 & 3 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & -2 & 2 & 2 \\ 0 & 1 & 3 & 3 \\ 0 & 5 & -1 & -3 \end{array}\right] \xrightarrow{\text{pivot on } a_{22}} \left[\begin{array}{ccc|c} 1 & 0 & 8 & 8 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & -16 & -18 \end{array}\right] \xrightarrow{\text{pivot on } a_{22}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \# \\ 0 & 1 & 0 & \# \\ 0 & 0 & 1 & \# \end{array}\right]$$

$\mathit{rank}\ A = \mathit{rank}\ \left[A \mid \vec{b}\right] = 3 \Rightarrow A\vec{x} = \vec{b}$ has a solution

Example

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & -3 & 4 \\ 2 & -1 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 1 & -3 & 4 & 5 \\ 2 & -1 & 7 & 6 \end{array}\right] \xrightarrow{\text{pivot on } a_{11}} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -5 & 1 & 1 \\ 0 & -5 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{-R_2 + R_3} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -5 & 1 & 1 \\ 0 & 0 & 0 & -3 \end{array}\right]$$

$\Rightarrow$ rank $A = 2$, rank $\left[A \mid \vec{b}\right] = 3$

$\Rightarrow A\vec{x} = \vec{b}$ has no solution

Tfae for *n* x *n A*:

1. A is nonsingular.
2. $A\vec{x} = \vec{0}$ has only the trivial solution.
3. A is row (col) equivalent to I_n.
4. For every $\vec{b} \in \mathbb{R}^n$, $A\vec{x} = \vec{b}$ has a unique solution.
5. A is a product of elementary matrices.
6. $\det(A) \neq 0$.
7. rank $A = n$
8. nullity $A = 0$
9. The rows of A form a linearly independent set of vectors in $\mathbb{R}_n$.
10. The columns of A form a linearly independent set of vectors in $\mathbb{R}^n$.