Math 270 Linear Algebra

Chapter 4 Real Vector Spaces

4.8 Coordinates and Isomorphism

$S=\{\vec{v}_1,\vec{v}_2,\ldots,\vec{v}_n\}$ **ordered basis** different from $S_1=\{\vec{v}_2,\vec{v}_1,\ldots,\vec{v}_n\}$

$\vec{v}\in V \Rightarrow \vec{v}=a_1\vec{v}_1+a_2\vec{v}_2+\ldots+a_n\vec{v}_n$.

Let $\left[\vec{v}\right]_S=\begin{bmatrix}a_1\\a_2\\ \\a_n\end{bmatrix}$. $\left[\vec{v}\right]_S$ is called the **coordinate vector of $\vec{v}$ with respect to the ordered basis *S***; the entries are called the **coordinates of $\vec{v}$ with respect to *S***.

Exercise #4 $V=P_2$

$S=\{t^2-t+1,\ t+1,\ t^2+1\}$

$T=\{t^2,\ t,\ 1\}$

$\vec{v}=4t^2-2t+3$

Compute $\left[\vec{v}\right]_T$ and $\left[\vec{v}\right]_S$.

Solution:

$$\vec{v}=4t^2-2t+3\Rightarrow\left[\vec{v}\right]_T=\begin{bmatrix}4\\-2\\3\end{bmatrix}$$

$$4t^2-2t+3=a_1\left(t^2-t+1\right)+a_2\left(t+1\right)+a_3\left(t^2+1\right)$$

$$\Rightarrow\begin{array}{rrrl}a_1 & & +a_3 & =4\\ -a_1 & +a_2 & & =-2\\ a_1 & +a_2 & +a_3 & =3\end{array}$$

$$\Rightarrow\left[\begin{array}{rrr|r}1&0&1&4\\-1&1&0&-2\\1&1&1&3\end{array}\right]\longrightarrow\left[\begin{array}{rrr|r}1&0&1&4\\0&1&1&2\\0&1&0&-1\end{array}\right]\longrightarrow\left[\begin{array}{rrr|r}1&0&1&4\\0&1&1&2\\0&0&-1&-3\end{array}\right]$$

$$\longrightarrow\left[\begin{array}{rrr|r}1&0&0&1\\0&1&0&-1\\0&0&1&3\end{array}\right]\Rightarrow\begin{array}{l}a_1=1\\a_2=-1\\a_3=3\end{array}\Rightarrow\left[\vec{v}\right]_S=\begin{bmatrix}1\\-1\\3\end{bmatrix}$$

Remark Let $S=\{\vec{v}_1,\vec{v}_2,\ldots,\vec{v}_n\}$ be an ordered basis for a vector space V. Let $\vec{v}$, $\vec{w}$ be vectors in V such that $\left[\vec{v}\right]_S=\left[\vec{w}\right]_S$. Then $\vec{v}=\vec{w}$.

Isomorphisms

Given a vector space V

an ordered basis $S=\{\vec{v}_1,\vec{v}_2,...,\vec{v}_n\}$ for V

$\vec{v}$ and $\vec{w}$ can be written uniquely as

$$\vec{v}=a_1\vec{v}_1+a_2\vec{v}_2+...+a_n\vec{v}_n$$
$$\vec{w}=b_1\vec{v}_1+b_2\vec{v}_2+...+b_n\vec{v}_n$$

Thus, we have the association

$$\vec{v}\to\left[\vec{v}\right]_S$$
$$\vec{w}\to\left[\vec{w}\right]_S$$

Since $\vec{v}+\vec{w}=(a_1+b_1)\vec{v}_1+(a_2+b_2)\vec{v}_2+...+(a_n+b_n)\vec{v}_n$, we can associate

$$\vec{v}+\vec{w}\to\left[\vec{v}+\vec{w}\right]_S=\left[\vec{v}\right]_S+\left[\vec{w}\right]_S.$$

Similarly, if c is a real number,

$$c\vec{v}=(ca_1)\vec{v}_1+(ca_2)\vec{v}_2+...+(ca_n)\vec{v}_n$$
$$\Rightarrow\left[c\vec{v}\right]_S=\begin{bmatrix}ca_1\\ca_2\\ \\ca_n\end{bmatrix}=c\left[\vec{v}\right]_S$$

and we can associate

$$c\vec{v}\to\left[c\vec{v}\right]_S=c\left[\vec{v}\right]_S.$$

The above suggests that algebraically, V and $\mathbb{R}^n$ behave “rather similarly”.

Definition Let V and W be vector spaces. $L:V\to W$ is **<u>one-to-one</u>** if

$$L(\vec{v}_1)=L(\vec{v}_2)\text{ for }\vec{v}_1,\vec{v}_2\in V\Rightarrow\vec{v}_1=\vec{v}_2.$$

L is **<u>onto</u>** if for each $\vec{w}\in W$ there is at least one $\vec{v}\in V$ for which $L(\vec{v})=\vec{w}$.

Example $L:\mathbb{R}^3 \to \mathbb{R}^2$

$$L\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\right) = \begin{bmatrix} a_1 + a_2 \\ a_1 \end{bmatrix}$$

L is <u>onto</u>: For $\vec{w} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \in \mathbb{R}^2$, find $\vec{v} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \mathbb{R}^3$ s.t. $L(\vec{v}) = \vec{w}$.

$$\begin{bmatrix} a_1 + a_2 \\ a_1 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \Rightarrow \begin{array}{c} a_1 = b_2 \\ a_2 = b_1 - b_2 \\ a_3 \text{ arbitrary} \end{array}$$

L is <u>NOT</u> 1 – 1: $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$; $L(\vec{v}_1) = L(\vec{v}_2)$ but $\vec{v}_1 \neq \vec{v}_2$

Definition Let $\langle V, \oplus, \odot \rangle$ and $\langle W, \boxplus, \boxdot \rangle$ be real vector spaces. A one-to-one and onto mapping $L:V \to W$ is called an <u>**isomorphism**</u> of V onto W if

(a) $L(\vec{v} \oplus \vec{w}) = L(\vec{v}) \boxplus L(\vec{w})$ for $\vec{v}, \vec{w} \in V$

(b) $L(c \odot \vec{v}) = c \boxdot L(\vec{v})$ for $\vec{v} \in V$ and $c \in \mathbb{R}$

In this case we say that <u>***V*** **is isomorphic to** ***W***</u> and we write $V \cong W$.

Note: If $L:V \to W$ is an isomorphism then

$$L(a_1 \odot \vec{v}_1 \oplus a_2 \odot \vec{v}_2 \oplus \ldots \oplus a_k \odot \vec{v}_k) = a_1 \boxdot L(\vec{v}_1) \boxplus a_2 \boxdot L(\vec{v}_2) \boxplus \ldots \boxplus a_k \boxdot L(\vec{v}_k)$$

where $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are vectors in V and $a_1, a_2, \ldots, a_k$ are scalars.

Remark $L:V \to W$ satisfying (a) and (b) is called a <u>**linear transformation**</u>. Thus an isomorphism is a linear transformation that is one-to-one and onto.

Theorem If V is a vector space and dim $V = n$ then $V \cong \mathbb{R}^n$.

Proof:

Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be an ordered basis for V. Define $L:V \to \mathbb{R}^n$ by

$$L(\vec{v}) = [\vec{v}]_S = \begin{bmatrix} a_1 \\ a_2 \\ \\ a_n \end{bmatrix}, \text{ where } \vec{v} = a_1\vec{v}_1 + a_2\vec{v}_2 + \ldots + a_n\vec{v}_n$$

Show that L is an isomorphism.

L is one-to-one:

$$\left[\vec{v}\right]_S = \begin{bmatrix} a_1 \\ a_2 \\ \\ a_n \end{bmatrix}, \quad \left[\vec{w}\right]_S = \begin{bmatrix} b_1 \\ b_2 \\ \\ b_n \end{bmatrix}$$

Suppose $L(\vec{v}) = L(\vec{w}) \Rightarrow \left[\vec{v}\right]_S = \left[\vec{w}\right]_S \Rightarrow \vec{v} = \vec{w}$.

L is onto:

If $\vec{w} = \begin{bmatrix} b_1 \\ b_2 \\ \\ b_n \end{bmatrix} \in \mathbb{R}^n$, let $\vec{v} = b_1\vec{v}_1 + b_2\vec{v}_2 + ... + b_n\vec{v}_n$.

Then $L(\vec{v}) = \vec{w}$.

(a): Let

$\vec{v},\ \vec{w} \in V$ s.t.

$\left[\vec{v}\right]_S = \begin{bmatrix} a_1 \\ a_2 \\ \\ a_n \end{bmatrix}$ and $\left[\vec{w}\right]_S = \begin{bmatrix} b_1 \\ b_2 \\ \\ b_n \end{bmatrix}$

Then $L(\vec{v} + \vec{w}) = \left[\vec{v} + \vec{w}\right]_S = \left[\vec{v}\right]_S + \left[\vec{w}\right]_S = L(\vec{v}) + L(\vec{w})$

(b): $L(c\vec{v}) = \left[c\vec{v}\right]_S = c\left[\vec{v}\right]_S = cL(\vec{v})$

Theorem Isomorphism is an equivalence relation, i.e.

(a) $V \cong V$ (**reflexive**)
(b) $V \cong W \Rightarrow W \cong V$ (**symmetric**)
(c) $V \cong W$ and $W \cong U \Rightarrow V \cong U$ (**transitive**)

Theorem Two finite-dimensional vector spaces are isomorphic if and only if their dimensions are equal.

Proof:

$("\Leftarrow")$ Let dim $V =$ dim W

$\Rightarrow V \cong \mathbb{R}^n$ and $W \cong \mathbb{R}^n$

$\Rightarrow V \cong W$

$("\Rightarrow")$ Let $V \cong W$.

Let $L: V \to W$ be an isomorphism.

Assume dim $V - n$ and let $S = \{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}$ be a basis for V.

To prove: $T = \{L(\vec{v}_1), L(\vec{v}_2), ..., L(\vec{v}_n)\}$ is a basis for *W*.

<u>*T* spans *W*</u>:

Since *L* is onto, $\vec{w} \in W \Rightarrow \vec{w} = L(\vec{v})$ for some $\vec{v} \in V$.

S is a basis for $V \Rightarrow \vec{v} = a_1\vec{v}_1 + a_2\vec{v}_2 + ... + a_n\vec{v}_n$ for unique real numbers a_i

$$\begin{aligned} L(\vec{v}) &= L(a_1\vec{v}_1 + a_2\vec{v}_2 + ... + a_n\vec{v}_n) \\ &= L(a_1\vec{v}_1) + L(a_2\vec{v}_2) + ... + L(a_n\vec{v}_n) \\ &= a_1L(\vec{v}_1) + a_2L(\vec{v}_2) + ... + a_nL(\vec{v}_n) \end{aligned}$$

<u>*T* is linearly independent</u>:

$$a_1L(\vec{v}_1) + a_2L(\vec{v}_2) + ... + a_nL(\vec{v}_n) = \vec{0}_W$$

By Exercise #29, $L(\vec{0}_V) = \vec{0}_W$.

Thus, $a_1\vec{v}_1 + a_2\vec{v}_2 + ... + a_n\vec{v}_n = \vec{0}_V$ since *L* is 1 – 1.

This implies $a_1 = a_2 = ... = a_n = 0$ since *S* is linearly independent.

Hence, *T* is linearly independent.

Thus, *T* is a basis for *W* and dim *W* = *n*.

Corollary If *V* is finite-dimensional and $V \cong \mathbb{R}^n$, then dim *V* = *n*.

Remarks

1. $L : V \to W$ is an isomorphism $\Rightarrow L^{-1} : W \to V$ exists and is also an isomorphism.
2. $S = \{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}$ basis for $V \Rightarrow T = L(S) = \{L(\vec{v}_1), L(\vec{v}_2), ..., L(\vec{v}_n)\}$ is a basis for *W*.

Note P_3, $\mathbb{R}^4$, and $\mathbb{R}_4$ are all isomorphic.

Transition Matrices

Given a vector space *V*, dim *V* = *n*

2 ordered bases $S = \{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}$ and $T = \{\vec{w}_1, \vec{w}_2, ..., \vec{w}_n\}$

$$\vec{v} \in V \Rightarrow \vec{v} = c_1\vec{w}_1 + c_2\vec{w}_2 + ... + c_n\vec{w}_n \Rightarrow \left[\vec{v}\right]_T = \begin{bmatrix} c_1 \\ c_2 \\ \\ c_n \end{bmatrix}$$

$$\begin{aligned} \left[\vec{v}\right]_S &= \left[c_1\vec{w}_1 + c_2\vec{w}_2 + ... + c_n\vec{w}_n\right]_S \\ &= \left[c_1\vec{w}_1\right]_S + \left[c_2\vec{w}_2\right]_S + ... + \left[c_n\vec{w}_n\right]_S \qquad (*) \\ &= c_1\left[\vec{w}_1\right]_S + c_2\left[\vec{w}_2\right]_S + ... + c_n\left[\vec{w}_n\right]_S \end{aligned}$$

Let $\left[\vec{w}_j\right]_S = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \\ a_{nj} \end{bmatrix}$.

The n x n matrix whose jth column is $\left[\vec{w}_j\right]_S$ is called the **<u>transition matrix from the *T* basis to the *S* basis</u>** and is denoted by $P_{S \leftarrow T}$.

Then (*) in matrix form is

$$\left[\vec{v}\right]_S = P\left[\vec{v}\right]_T .$$

Diagram

$$\vec{v} \in V \longrightarrow \left[\vec{v}\right]_S \in \mathbb{R}^n$$

$$\downarrow \qquad \nearrow$$

$$\downarrow \qquad \nearrow \quad P_{S \leftarrow T}\left[\vec{v}\right]_T$$

$$\left[\vec{v}\right]_T \in \mathbb{R}^n \nearrow$$

Exercise #16 $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}, \quad T = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ ordered bases for $\mathbb{R}^3$

$\vec{v} = \begin{bmatrix} 1 \\ 3 \\ 8 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} -1 \\ 8 \\ -2 \end{bmatrix}$

(a) Find $\left[\vec{v}\right]_T$ and $\left[\vec{w}\right]_T$.

Solution:

$$\left[\begin{array}{ccc|c|c} -1 & 1 & 0 & 1 & -1 \\ 1 & 2 & 1 & 3 & 8 \\ 0 & -1 & 0 & 8 & -2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c|c} 1 & -1 & 0 & -1 & 1 \\ 0 & 3 & 1 & 4 & 7 \\ 0 & -1 & 0 & 8 & -2 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c|c} 1 & -1 & 0 & -1 & 1 \\ 0 & -1 & 0 & 8 & -2 \\ 0 & 3 & 1 & 4 & 7 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c|c} 1 & 0 & 0 & -9 & 3 \\ 0 & 1 & 0 & -8 & 2 \\ 0 & 0 & 1 & 28 & 1 \end{array}\right]$$

$$\Rightarrow \left[\vec{v}\right]_T = \begin{bmatrix} -9 \\ -8 \\ 28 \end{bmatrix}, \quad \left[\vec{w}\right]_T = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

(b) Find $P_{S\leftarrow T}$

Solution:

To find $a_1,\ a_2,\ a_3$ s.t. $a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 = \vec{w}_1$.

Similarly…

$$b_1\vec{v}_1 + b_2\vec{v}_2 + b_3\vec{v}_3 = \vec{w}_2$$
$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{w}_3$$

Solve $\left[\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3 \mid \vec{w}_1 \mid \vec{w}_2 \mid \vec{w}_3\right]$.

$$\left[\begin{array}{ccc|c|c|c} 1 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 2 & 1 \\ 1 & 0 & 2 & 0 & -1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c|c|c} 1 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 2 & 1 \\ 0 & 1 & 2 & 1 & -2 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c|c|c} 1 & -1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & 1 & -2 & 0 \\ 0 & 0 & 1 & 1 & 2 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c|c|c} 1 & 0 & 2 & 0 & -1 & 0 \\ 0 & 1 & 2 & 1 & -2 & 0 \\ 0 & 0 & 1 & 1 & 2 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c|c|c} 1 & 0 & 0 & -2 & -5 & -2 \\ 0 & 1 & 0 & -1 & -6 & -2 \\ 0 & 0 & 1 & 1 & 2 & 1 \end{array}\right] \Rightarrow P_{S\leftarrow T} = \begin{bmatrix} -2 & -5 & -2 \\ -1 & -6 & -2 \\ 1 & 2 & 1 \end{bmatrix}$$

(c) Find $\left[\vec{v}\right]_S$ and $\left[\vec{w}\right]_S$ using $P_{S\leftarrow T}$.

Solution:

$$\left[\vec{v}\right]_S = P_{S\leftarrow T}\left[\vec{v}\right]_T = \begin{bmatrix} -2 & -5 & -2 \\ -1 & -6 & -2 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} -9 \\ -8 \\ 28 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

$$\left[\vec{w}\right]_S = P_{S\leftarrow T}\left[\vec{w}\right]_T = \begin{bmatrix} -2 & -5 & -2 \\ -1 & -6 & -2 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -18 \\ -17 \\ 8 \end{bmatrix}$$

(d) Find $\left[\vec{v}\right]_S$ and $\left[\vec{w}\right]_S$ directly.

Solution:

$$\left[\begin{array}{ccc|cc} 1 & -1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 3 & 8 \\ 1 & 0 & 2 & 8 & -2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|cc} 1 & -1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 3 & 8 \\ 0 & 1 & 2 & 7 & -1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|cc} 1 & -1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 3 & 8 \\ 0 & 1 & 0 & 1 & -17 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 2 & -18 \\ 0 & 0 & 1 & 3 & 8 \\ 0 & 1 & 0 & 1 & -17 \end{array}\right]$$

(e) Find $Q_{T\leftarrow S}$

Solution:

$$\left[\vec{w}_1 \quad \vec{w}_2 \quad \vec{w}_3 \;\middle|\; \vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3\right]$$

$$\left[\begin{array}{ccc|ccc} -1 & 1 & 0 & 1 & -1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 & 0 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & -1 & 1 & 0 \\ 0 & 3 & 1 & 1 & -1 & 1 \\ 0 & -1 & 0 & 1 & 0 & 2 \end{array}\right]$$

$$\xrightarrow{\text{pivot on } a_{32}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 1 & -2 \\ 0 & 0 & 1 & 4 & -1 & 7 \\ 0 & 1 & 0 & -1 & 0 & -2 \end{array}\right] \Rightarrow Q_{T\leftarrow S} = \begin{bmatrix} -2 & 1 & -2 \\ -1 & 0 & -2 \\ 4 & -1 & 7 \end{bmatrix}$$

(f) Find $\left[\vec{v}\right]_T$ and $\left[\vec{w}\right]_T$ using $Q_{T\leftarrow S}$

Solution:

$$\left[\vec{v}\right]_T = Q_{T\leftarrow S}\left[\vec{v}\right]_S = \begin{bmatrix} -2 & 1 & -2 \\ -1 & 0 & -2 \\ 4 & -1 & 7 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -9 \\ -8 \\ 28 \end{bmatrix}$$

$$\left[\vec{w}\right]_T = Q_{T\leftarrow S}\left[\vec{w}\right]_S = \begin{bmatrix} -2 & 1 & -2 \\ -1 & 0 & -2 \\ 4 & -1 & 7 \end{bmatrix}\begin{bmatrix} -18 \\ -17 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

Note: $P_{S\leftarrow T} \cdot Q_{T\leftarrow S}$

$$= \begin{bmatrix} -2 & -5 & -2 \\ -1 & -6 & -2 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} -2 & 1 & -2 \\ -1 & 0 & -2 \\ 4 & -1 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3 \Rightarrow Q_{T\leftarrow S} = P_{S\leftarrow T}^{-1}$$