Math 270 Linear Algebra

Chapter 4 Real Vector Spaces

4.7 Homogeneous Systems

Goal: To find a basis for the solution space of a homogeneous system $A\vec{x} = \vec{0}$.

Exercise #6

$$\left[\begin{array}{ccccc|c} 1 & -1 & 2 & 3 & 4 & 0 \\ -1 & 2 & 3 & 4 & 5 & 0 \\ 1 & -1 & 3 & 5 & 6 & 0 \\ 3 & -4 & 1 & 2 & 3 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 1 & -1 & 2 & 3 & 4 & 0 \\ 0 & 1 & 5 & 7 & 9 & 0 \\ 0 & 0 & 1 & 2 & 2 & 0 \\ 0 & -1 & -5 & -7 & -9 & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 7 & 10 & 13 & 0 \\ 0 & 1 & 5 & 7 & 9 & 0 \\ 0 & 0 & 1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & -4 & -1 & 0 \\ 0 & 1 & 0 & -3 & -1 & 0 \\ 0 & 0 & 1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \vec{x} = \begin{bmatrix} 4s+t \\ 3s+t \\ 2s+2t \\ s \\ t \end{bmatrix},\ s,t \text{ real} \Rightarrow \vec{x} = s\begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

Let $s = 1$ and $t = 0 \Rightarrow \vec{x}_1 = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}$. Let $s = 0$ and $t = 1 \Rightarrow \vec{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$.

Thus, $\{\vec{x}_1, \vec{x}_2\}$ is a basis for W and dim $W = 2$.

Exercise #14 Find a basis for the solution space of the homogeneous system $(\lambda I_n - A)\vec{x} = \vec{0}$ for $\lambda = -3$ and $A = \begin{bmatrix} -4 & -3 \\ 2 & 3 \end{bmatrix}$.

Solution:

$$-3I_2 - A = -3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -4 & -3 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -2 & -6 \end{bmatrix}$$

Solve:

$$\left[\begin{array}{cc|c} 1 & 3 & 0 \\ -2 & -6 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \vec{x} = \begin{bmatrix} -3s \\ s \end{bmatrix},\ s \text{ real} \Rightarrow = \vec{x} = s\begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

$$\Rightarrow \left\{\begin{bmatrix} -3 \\ 1 \end{bmatrix}\right\} \text{ is a basis for the solution space}$$

Exercise #4 Find a basis for and the dimension of the solution space of

$$\begin{bmatrix} 1 & -1 & 1 & -2 & 1 \\ 3 & -3 & 2 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Solution:

$$\left[\begin{array}{ccccc|c} 1 & -1 & 1 & -2 & 1 & 0 \\ 3 & -3 & 2 & 0 & 2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 1 & -1 & 1 & -2 & 1 & 0 \\ 0 & 0 & -1 & -6 & -1 & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccccc|c} 1 & -1 & 0 & 4 & 0 & 0 \\ 0 & 0 & 1 & -6 & 1 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 = x_2 - 4x_4 \\ x_3 = 6x_4 - x_5 \end{array}$$

$$\Rightarrow \begin{array}{l} x_1 = r - 4s \\ x_2 = r \\ x_3 = 6r - t \\ x_4 = s \\ x_5 = t \end{array} \Rightarrow \begin{bmatrix} r & -4s & \\ r & & \\ 6r & & -t \\ & s & \\ & & t \end{bmatrix} = r\begin{bmatrix} 1 \\ 1 \\ 6 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \text{basis for the solution space is } \left\{\begin{bmatrix} 1 \\ 1 \\ 6 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}\right\}$$

dim or **<u>nullity</u>** of A is 3.

Exercise #18 Find all real numbers λ such that $(\lambda I_n - A)\vec{x} = \vec{0}$ has a nontrivial solution.

$$A = \begin{bmatrix} 3 & 0 \\ 2 & -2 \end{bmatrix}.$$

Solution:

$$\lambda I_2 - A = \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} \lambda - 3 & 0 \\ -2 & \lambda + 2 \end{bmatrix}$$

$(\lambda I_2 - A)\vec{x} = \vec{0}$:

$$\left[\begin{array}{cc|c} \lambda - 3 & 0 & 0 \\ -2 & \lambda + 2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & \lambda + 2 & 0 \end{array}\right]$$

will have a nontrivial solution if $\lambda + 2 = 0$ or $\lambda = -2$

Consider the systems

$$A\vec{x} = \vec{b} \qquad \text{(a)}$$

and

$$A\vec{x} = \vec{0} \qquad \text{(b)}$$

Solutions to (a) can be written as $\vec{x} = \vec{x}_p + \vec{x}_h$

where $\vec{x}_p =$ particular solution to (a)
$\vec{x}_h =$ particular solution to (b)

Exercise #22

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 2 \\ 1 & -3 & 2 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$$

Solution:

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 1 & 1 & 2 & -1 \\ 1 & -3 & 2 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 2 & 0 & -2 \\ 0 & -2 & 0 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \vec{x} = \begin{bmatrix} -2s \\ -1 \\ s \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}}_{\vec{x}_p} + s\underbrace{\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}}_{\vec{x}_h}$$