Math 270 Linear Algebra

Chapter 4 Real Vector Spaces

4.6 Basis and Dimension

Definition The vectors $\vec{v}_1, \vec{v}_2, ..., \vec{v}_k$ in a vector space V are said to form a **basis** for V if

(a) $\vec{v}_1, \vec{v}_2, ..., \vec{v}_k$ span V, and

(b) $\vec{v}_1, \vec{v}_2, ..., \vec{v}_k$ are linearly independent

Remark If $\vec{v}_1, \vec{v}_2, ..., \vec{v}_k$ form a basis for a vector space V, then $\vec{v}_1, \vec{v}_2, ..., \vec{v}_k$ must be distinct and nonzero.

Example 1 $V = \mathbb{R}^3$

$$\vec{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$ **natural basis** or **standard basis** for $\mathbb{R}^3$

$$\vec{v} \in \mathbb{R}^3, \; \vec{v} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$$

Basis for $V = \mathbb{R}^n$ is $\{\vec{e}_1, \vec{e}_2, ..., \vec{e}_n\}$, where $\vec{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i\text{th row}$

Exercise #2 Which are bases for $\mathbb{R}^3$

(a) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$

Solution:

(i) Does the set span $\mathbb{R}^3$?

Take any $\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3$. Can we find constants a_1, a_2 s.t.

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = a_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} ?$$

$$\left[\begin{array}{cc|c} 1 & 0 & a \\ 2 & 1 & b \\ 0 & -1 & c \end{array}\right] \Rightarrow 2a_1 + a_2 = b \Rightarrow b = 2a - c \Rightarrow \text{NO, since this gives a restriction on } b$$

$$a_1 = a, \quad a_2 = -c$$

(ii) no need to check

(b) $\left\{ \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

Solution:

(i) Does the set span $\mathbb{R}^3$?

$$\left[\begin{array}{ccc|c} 3 & -1 & 0 & a \\ 2 & 2 & 1 & b \\ 2 & 1 & 0 & c \end{array}\right] \xrightarrow{-R_2+R_1} \left[\begin{array}{ccc|c} 1 & -3 & -1 & a-b \\ 2 & 2 & 1 & b \\ 2 & 1 & 0 & c \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & -3 & -1 & a-b \\ 0 & 8 & 3 & -2a+3b \\ 0 & 7 & 2 & -2a+2b+c \end{array}\right] \xrightarrow{-R_3+R_2} \left[\begin{array}{ccc|c} 1 & -3 & -1 & a-b \\ 0 & 1 & 1 & b-c \\ 0 & 7 & 2 & -2a+2b+c \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & a+2b-3c \\ 0 & 1 & 1 & b-c \\ 0 & 0 & -5 & -2a-5b+9c \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & \text{something} \\ 0 & 1 & 0 & \text{something} \\ 0 & 0 & 1 & \frac{-2a-5b+9c}{-5} \end{array}\right]$$

$\Rightarrow$ YES, it spans $\mathbb{R}^3$

(ii) Is the set linearly independent?

Check whether $a_1 \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix} + a_2 \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + a_3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow a_1 = a_2 = a_3 = 0$.

$$\left[\begin{array}{ccc|c} 3 & -1 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 2 & 1 & 0 & 0 \end{array}\right] \xrightarrow{-R_2+R_1} \left[\begin{array}{ccc|c} 1 & -3 & -1 & 0 \\ 2 & 2 & 1 & 0 \\ 2 & 1 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -3 & -1 & 0 \\ 0 & 8 & 3 & 0 \\ 0 & 7 & 2 & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & -3 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 7 & 2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -5 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$\Rightarrow a_1 = a_2 = a_3 = 0 \Rightarrow$ YES, the set is linearly independent

Thus, (i) and (ii) imply that the set is a basis for $\mathbb{R}^3$.

Note:

1. We could have shortened the solution by solving:

$$\left[\begin{array}{ccc|c|c} 3 & -1 & 0 & a & 0 \\ 2 & 2 & 1 & b & 0 \\ 2 & 1 & 0 & c & 0 \end{array}\right].$$

2. From (i) we note that we were able to reduce A to I_3 which means A is nonsingular which implies $A\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \vec{0}$ has only the trivial solution which implies that we did not have to do (ii).

Example 2 Show that $S = \{t^2+1,\ t-1,\ 2t+2\}$ is a basis for P_2.

Solution:

(a) Does S span P_2?

Can we find constants $a_1,\ a_2,\ a_3$ s.t. for $at^2+bt+c \in P_2$ we have

$$at^2+bt+c = a_1\left(t^2+1\right)+a_2\left(t-1\right)+a_3\left(2t+2\right)$$

$$\begin{aligned} a &= a_1 \\ \Rightarrow \quad b &= a_2+2a_3 \\ c &= a_1-a_2+2a_3 \end{aligned}$$

(b) Is S linearly independent?

Solve:

$$\begin{aligned} a_1 &= 0 \\ a_2+2a_3 &= 0 \\ a_1-a_2+2a_3 &= 0 \end{aligned}$$

Thus, solve:

$$\left[\begin{array}{ccc|c|c} 1 & 0 & 0 & a & 0 \\ 0 & 1 & 2 & b & 0 \\ 1 & -1 & 2 & c & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c|c} 1 & 0 & 0 & a & 0 \\ 0 & 1 & 2 & b & 0 \\ 0 & -1 & 2 & c-a & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c|c} 1 & 0 & 0 & a & 0 \\ 0 & 1 & 2 & b & 0 \\ 0 & 0 & 4 & c-a+b & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c|c} 1 & 0 & 0 & a & 0 \\ 0 & 1 & 0 & * & 0 \\ 0 & 0 & 1 & \dfrac{c-a+b}{4} & 0 \end{array}\right] \Rightarrow S \text{ is a basis for } P_2$$

Remark $\{t^n, t^{n-1}, \ldots, t, 1\}$ is called the **natural**, or **standard basis** for P_n.

Example 4 Find a basis for the subspace V of P_2 consisting of all vectors of the form at^2+bt+c, where $c = a-b$.

Solution:

$v \in P_2$ looks like

$$at^2+bt+a-b = a\left(t^2+1\right)+b\left(t-1\right)$$

$$\Rightarrow \{t^2+1, t-1\} \text{ spans } V$$

To show $\{t^2+1, t-1\}$ is linearly independent:

$$
\begin{aligned}
& a_1\left(t^2+1\right)+a_2\left(t-1\right)=0 \\
\Rightarrow\ & a_1=0 \\
& a_2=0 \\
& a_1-a_2=0 \\
\Rightarrow\ & a_1=a_2=0
\end{aligned}
$$

Thus, $\left\{t^2+1, t-1\right\}$ is a basis for V.

Exercise #20

a) Find a basis for the subspace of $\mathbb{R}^3$ consisting of all vectors of the form $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, where $a=0$

Solution:

$\begin{bmatrix} 0 \\ b \\ c \end{bmatrix}=b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}+c\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\Rightarrow\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ span the subspace.

Obviously, the vectors are linearly independent and thus form a basis.

b) $V \subseteq \mathbb{R}^4$

$\vec{v}\in V$ looks like $\begin{bmatrix} a+c \\ a-b \\ b+c \\ -a+b \end{bmatrix}$

Solution:

$$
\begin{bmatrix} a+c \\ a-b \\ b+c \\ -a+b \end{bmatrix}=\begin{bmatrix} a & & +c \\ a & -b & \\ & b & c \\ -a & b & \end{bmatrix}=a\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}+b\begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix}+c\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}
$$

$\Rightarrow\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix},\begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ span V.

To show linear independence:

$$
\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 1 & 0 & 0 \end{array}\right]\longrightarrow\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]\longrightarrow\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]
$$

$$
\begin{aligned}
& a_1=-r \\
\Rightarrow\ & a_2=-r,\ r \text{ real} \Rightarrow \text{NOT linearly independent} \\
& a_3=r
\end{aligned}
$$

Note: $\begin{bmatrix}1\\0\\1\\0\end{bmatrix}=\begin{bmatrix}1\\1\\0\\-1\end{bmatrix}+\begin{bmatrix}0\\-1\\1\\1\end{bmatrix}.\ \vec{v}_1$

Thus, $\left\{\begin{bmatrix}1\\1\\0\\-1\end{bmatrix},\begin{bmatrix}0\\-1\\1\\1\end{bmatrix}\right\}$ is a basis for V.

Definition A vector space V is called **finite-dimensional** if there is a finite subset of V that is a basis for V; otherwise, it is **infinite-dimensional**.

Note: $\{\vec{v}_1,\vec{v}_2,...,\vec{v}_k\}$ is a basis for a vector space $V \Rightarrow \{c\vec{v}_1,\vec{v}_2,...,\vec{v}_k\}$ is a basis when $c \neq 0 \Rightarrow$ a basis for a vector space is never unique,

Theorem $S=\{\vec{v}_1,\vec{v}_2,...,\vec{v}_n\}$ is a basis for a vector space $V \Rightarrow \vec{v} \in V$ can be written in one and only one way as a linear combination of the vectors in S.

Proof:

S spans $V \Rightarrow \vec{v} \in V$ can be written as $\vec{v}=a_1\vec{v}_1+a_2\vec{v}_2+...+a_n\vec{v}_n$.

Suppose $\vec{v}$ can also be written as $\vec{v}=b_1\vec{v}_1+b_2\vec{v}_2+...+b_n\vec{v}_n$.

$\vec{0}=\vec{v}-\vec{v}=(a_1-b_1)\vec{v}_1+(a_2-b_2)\vec{v}_2+...+(a_n-b_n)\vec{v}_n$

Then $\Rightarrow a_i-b_i=0$ for all i since S is linearly independent

$\Rightarrow a_i=b_i \quad \forall i$

Remark Exercise 44 shows that the converse also holds.

Theorem Let $S=\{\vec{v}_1,\vec{v}_2,...,\vec{v}_n\}$ be a set of nonzero vectors in a vector space V, and let $W=span\ S$. Then some subset of S is a basis for W.

Proof:

Case 1: S is linearly independent.
We're done since S already spans W.

Case 2: S is linearly dependent

$\Rightarrow a_1\vec{v}_1+a_2\vec{v}_2+...+a_n\vec{v}_n=\vec{0}$, where not all a_i are 0

$\Rightarrow$ some $\vec{v}_j$ is a linear combination of the preceding vectors and we can delete $\vec{v}_j$ and $S_1=\{\vec{v}_1,\vec{v}_2,...,\vec{v}_{j-1},\vec{v}_{j+1},...,\vec{v}_n\}$ also spans W.

If S_1 is linearly independent we are done, otherwise, repeat the process until we get a basis T for W.

The proof gives a simple procedure for finding a subset *T* of a set *S* so that *T* is a basis for span *S*. Let $S = \{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}$ be a set of nonzero vectors in *V*.

<u>Step 1</u> Form $a_1\vec{v}_1 + a_2\vec{v}_2 + ... + a_n\vec{v}_n = \vec{0}$ and solve for $a_1, a_2, ..., a_n$. If all are zero, then *S* is linearly independent and *S* is a basis for *W*.

<u>Step 2</u> Otherwise, one of the vectors, say $\vec{v}_j$, is a linear combination of the preceding vectors. Delete $\vec{v}_j$ to form S_1.

<u>Step 3</u> Repeat Step 1 using S_1.

If $V = \mathbb{R}^m$ or $\mathbb{R}_m$, do the following:

<u>Step 1</u> Form

$$a_1\vec{v}_1 + a_2\vec{v}_2 + ... + a_n\vec{v}_n = \vec{0} \qquad (1)$$

<u>Step 2</u> Construct the augmented matrix associated with (1) and transform r.r.e.f.

<u>Step 3</u> The vectors corresponding to the columns containing the leading 1's form a basis *T* for $W = span\ S$.

Example 5 $V = \mathbb{R}_3$, $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5\}$, where

$$\vec{v}_1 = [1 \ \ 0 \ \ 1], \vec{v}_2 = [0 \ \ 1 \ \ 1], \vec{v}_3 = [1 \ \ 1 \ \ 2]$$

$$\vec{v}_4 = [1 \ \ 2 \ \ 1], \vec{v}_5 = [-1 \ \ 1 \ \ -2]$$

Solution:

S spans $\mathbb{R}_3$. (Verify.)

To find a subset of *S* that is a basis for $\mathbb{R}_3$.

<u>Step 1</u>: $a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 + a_4\vec{v}_4 + a_5\vec{v}_5 = \vec{0}$

$$a_1[1 \ \ 0 \ \ 1] + a_2[0 \ \ 1 \ \ 1] + a_3[1 \ \ 1 \ \ 2]$$
$$+a_4[1 \ \ 2 \ \ 1] + a_5[-1 \ \ 1 \ \ -2] = [0 \ \ 0 \ \ 0]$$

<u>Step 2</u>

$$\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 \\ 1 & 1 & 2 & 1 & -2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & -2 & -2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

$\Rightarrow \{\vec{v}_1, \vec{v}_2, \vec{v}_4\}$ is a basis for $\mathbb{R}_3$

Why?

The solution is

$$\begin{aligned} a_1 &= -a_3 + 2a_5 \\ a_2 &= -a_3 + a_5 \\ a_4 &= -a_5 \\ a_3, a_5 &\text{ free} \end{aligned}$$

Let $a_3 = 1,\ a_5 = 0$. Then

$$\begin{aligned} a_1 &= -1 \\ a_2 &= -1 \\ a_4 &= 0 \end{aligned}$$

$$-1\vec{v}_1 - 1\vec{v}_2 + \vec{v}_3 + 0\vec{v}_4 + 0\vec{v}_5 = 0 \Rightarrow \vec{v}_3 = \vec{v}_1 + \vec{v}_2 \text{ delete } \vec{v}_3$$

Let $a_3 = 0,\ a_5 = 1$. Then

$$\begin{aligned} a_1 &= 2 \\ a_2 &= 1 \\ a_4 &= -1 \end{aligned}$$

$$2\vec{v}_1 + \vec{v}_2 + 0\vec{v}_3 - \vec{v}_4 + \vec{v}_5 = 0 \Rightarrow \vec{v}_5 = -2\vec{v}_1 - \vec{v}_2 - \vec{v}_5 \text{ delete } \vec{v}_5$$

Thus, $\left\{\vec{v}_1, \vec{v}_2, \vec{v}_4\right\}$ spans $\mathbb{R}_3$.

Is $\left\{\vec{v}_1, \vec{v}_2, \vec{v}_4\right\}$ linearly independent?

Consider the matrix with columns $\vec{v}_1, \vec{v}_2, \vec{v}_4$:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{\cdots} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$ has only the trivial solution, then $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right]$ also has only the trivial solution.

Thus, $\left\{\vec{v}_1, \vec{v}_2, \vec{v}_4\right\}$ is linearly independent.

Exercise #12 Find a basis for the subspace W of $\mathbb{R}_4$ spanned by the set of vectors

$$\{[1\ \ 1\ \ 0\ \ -1],[0\ \ 1\ \ 2\ \ 1],[1\ \ 0\ \ 1\ \ -1],[1\ \ 1\ \ -6\ \ -3],[-1\ \ -5\ \ 1\ \ 0]\}.$$

Solution:

$$a_1[1\ \ 1\ \ 0\ \ -1]+a_2[0\ \ 1\ \ 2\ \ 1]+a_3[1\ \ 0\ \ 1\ \ -1]$$
$$+a_4[1\ \ 1\ \ -6\ \ -3]+a_5[-1\ \ -5\ \ 1\ \ 0]=[0\ \ 0\ \ 0\ \ 0]$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 1 & -1 & 0 \\ 1 & 1 & 0 & 1 & -5 & 0 \\ 0 & 2 & 1 & -6 & 1 & 0 \\ -1 & 1 & -1 & -3 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & -4 & 0 \\ 0 & 2 & 1 & -6 & 1 & 0 \\ 0 & 1 & 0 & -2 & -1 & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & -4 & 0 \\ 0 & 0 & 3 & -6 & 9 & 0 \\ 0 & 0 & 1 & -2 & 3 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 3 & -4 & 0 \\ 0 & 1 & 0 & -2 & -1 & 0 \\ 0 & 0 & 1 & -2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Thus, a basis is $\{[1\ \ 1\ \ 0\ \ -1],[0\ \ 1\ \ 2\ \ 1],[1\ \ 0\ \ 1\ \ -1]\}$.

Theorem If $S=\{\vec{v}_1,\vec{v}_2,\ldots,\vec{v}_n\}$ is a basis for a vector space V and $T=\{\vec{w}_1,\vec{w}_2,\ldots,\vec{w}_r\}$ is a linearly independent set of vectors in V, then $r \le n$.

Corollary If $S=\{\vec{v}_1,\vec{v}_2,\ldots,\vec{v}_n\}$ and $T=\{\vec{w}_1,\vec{w}_2,\ldots,\vec{w}_m\}$ are bases for a vector space V, then $n=m$.

Definition The **<u>dimension</u>** of a nonzero vector space V is the number of vectors in a basis.

Notation: dim V

dim $\{\vec{0}\}=0$

In Exercise 12, dim W = 3.

Definition Let S be a set of vectors in a vector space V. A subset T of S is called a **<u>maximal independent subset</u>** of S if T is a linearly independent set of vectors which is not properly contained in any other linearly independent subset of S.

Example 8 $V=\mathbb{R}^3$, $S=\{\vec{v}_1,\vec{v}_2,\vec{v}_3,\vec{v}_4\}$

$$\vec{v}_1=\begin{bmatrix}1\\0\\0\end{bmatrix},\ \vec{v}_2=\begin{bmatrix}0\\1\\0\end{bmatrix},\ \vec{v}_3=\begin{bmatrix}0\\0\\1\end{bmatrix},\ \vec{v}_4=\begin{bmatrix}1\\1\\1\end{bmatrix}$$

maximal independent subsets of S are $\{\vec{v}_1,\vec{v}_2,\vec{v}_3\}$, $\{\vec{v}_1,\vec{v}_2,\vec{v}_4\}$, $\{\vec{v}_1,\vec{v}_3,\vec{v}_4\}$, $\{\vec{v}_2,\vec{v}_3,\vec{v}_4\}$

Corollary dim $V = n \Rightarrow$ maximal independent subset of vectors in V contains n vectors

Definition If S is a set of vectors spanning a vector space V, then S is called a **<u>minimal spanning set</u>** for V if S does not properly contain any other set spanning V.

Corollary dim $V = n \Rightarrow$ minimal spanning set for V contains n vectors

Corollary dim $V = n \Rightarrow$ any subset of $m > n$ vectors must be linearly dependent

Corollary dim $V = n \Rightarrow$ any subset of $m < n$ vectors cannot span V

To construct a basis containing a linearly independent set of vectors:

Theorem If S is a linearly independent set of vectors in a finite-dimensional vector space V, then there is a basis T for V that contains S.

Proof:

Let $S = \left\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\right\}$ be linearly independent. Let dim $V = n$ and let $m < n$.

Let $\left\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n\right\}$ be a basis for V.

Let $S_1 = \left\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m, \vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n\right\}$.

S_1 spans $V \Rightarrow S_1$ contains a basis T for V and T is obtained by deleting from S_1 every vector that is a linear combination of the preceding vectors. Since S is linearly independent, none of the $\vec{v}_i$ will be deleted. Hence, T will contain S.

Exercise #28 Find a basis for $\mathbb{R}^3$ that includes

a) $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ **b)** $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$

Solution:

a) Form $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} = \left\{ \vec{v}_1, \underbrace{\vec{e}_1, \vec{e}_2, \vec{e}_3}_{\text{natural basis}} \right\}$.

Form $a_1\vec{v}_1 + a_2\vec{e}_1 + a_3\vec{e}_2 + a_4\vec{e}_3 = \vec{0}$.

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{2} & 0 \end{array}\right]$$

$\Rightarrow$ basis is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

b) Form $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

$$\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 2 & 3 & 0 & 0 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 3 & -2 & 0 & 1 & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & -3 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & -3/2 & 1/2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3/2 & -1/2 & 0 \end{array}\right]$$

Thus, the basis is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$.

Remark If W is a subspace of a finite-dimensional vector space V, then dim $W \leq$ dim V.

Theorem Let dim $V = n$.

(a) $S = \{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}$ linearly independent $\Rightarrow$ S is a basis for V.

(b) $S = \{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}$ spans $V \Rightarrow$ S is a basis for V.

Note: The theorem says that if we know that the dimension of V is n, we need to check only one of the conditions for a set with n vectors to be a basis.

Theorem Let S be a finite subset of the vector space V that spans V. A maximal independent subset T of S is a basis for V.