Math 270 Linear Algebra

Chapter 4 Real Vector Spaces

4.5 Linear Independence

Definition The vectors $\vec{v}_1, \vec{v}_2, ..., \vec{v}_k$ in a vector space V are said to be **linearly dependent** if there exist constants $a_1, a_2, ..., a_k$, not all zero, s.t.

$$a_1\vec{v}_1 + a_2\vec{v}_2 + ... + a_k\vec{v}_k = \vec{0}.$$

Otherwise, $\vec{v}_1, \vec{v}_2, ..., \vec{v}_k$ are said to be **linearly independent**. That is, for linear independence:

$$a_1\vec{v}_1 + a_2\vec{v}_2 + ... + a_k\vec{v}_k = \vec{0} \Rightarrow a_1 = a_2 = ... = a_k = 0.$$

If $S = \{\vec{v}_1, \vec{v}_2, ..., \vec{v}_k\}$, we say S is linearly independent/dependent if the vectors have the corresponding property.

Exercise 9 Let $\vec{x}_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$, $\vec{x}_2 = \begin{bmatrix} 4 \\ -7 \\ -1 \end{bmatrix}$, $\vec{x}_3 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ belong to the solution space of $A\vec{x} = \vec{0}$. Is $\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ linearly independent?

Solution:

$$a_1\vec{x}_1 + a_2\vec{x}_2 + a_3\vec{x}_3 = \vec{0}$$

$$a_1\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + a_2\begin{bmatrix} 4 \\ -7 \\ -1 \end{bmatrix} + a_3\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 2 & 4 & 1 & 0 \\ -1 & -7 & 2 & 0 \\ 1 & -1 & 2 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ -1 & -7 & 2 & 0 \\ 2 & 4 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & -8 & 4 & 0 \\ 0 & 6 & -3 & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3/2 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{c} a_1 = -\frac{3}{2}r \\ a_2 = \frac{1}{2}r \\ a_3 = r,\ r \text{ real} \end{array}$$

For example, if $r = 2$, then $a_1 = -3,\ a_2 = 1,\ a_3 = 2,$ not all zero.
Thus, the set is linearly dependent.

We can also use determinants to check linear independence of a set of n vectors in $\mathbb{R}^n$ or $\mathbb{R}_n$:

Theorem Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a set of n vectors in $\mathbb{R}^n$ $(\mathbb{R}_n)$. Let A be the matrix whose columns (rows) are elements of S. Then S is linearly independent if and only if $\det(A) \neq 0$.

Proof for $\mathbb{R}^n$:

("$\Rightarrow$") S linearly independent $\Rightarrow$ r.r.e.f. of A is I_n
$\Rightarrow$ A is row equivalent to I_n
$\Rightarrow \det(A) \neq 0$

("$\Leftarrow$") $\det(A) \neq 0 \Rightarrow A$ is row equivalent to I_n
Suppose S is linearly dependent. Then the r.r.e.f. of A will have a zero row, a contradiction.

Another Solution to Exercise 9

$$\left|\begin{array}{ccc} 2 & 4 & 1 \\ -1 & -7 & 2 \\ 1 & -1 & 2 \end{array}\right| \begin{array}{cc} 2 & 4 \\ -1 & -7 \\ 1 & -1 \end{array} = -28 + 8 + 1 - (-7 - 4 - 8) = 0$$

$\Rightarrow$ the set is linearly dependent

Theorem Let S_1 and S_2 be finite subsets of a vector space and let S_1 be a subset of S_2. Then

(a) S_1 linearly dependent $\Rightarrow S_2$ linearly dependent
(b) S_2 linearly independent $\Rightarrow S_1$ linearly independent

Proof:

Let $S_1 = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$
$S_2 = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k, \vec{v}_{k+1}, \ldots, \vec{v}_m\}$

S_1 linearly dependent $\Rightarrow$ there exist $a_1, a_2, \ldots, a_k$, not all zero, s.t.

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \ldots + a_k\vec{v}_k = \vec{0}.$$

Thus,

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \ldots + a_k\vec{v}_k + 0\vec{v}_{k+1} + \ldots + 0\vec{v}_m = \vec{0}.$$

Since not all a_i are 0, S_2 is linearly dependent. This proves (a).

Since (b) is the contrapositive of (a) we don't need to prove it.

Remarks

1) $S=\{\vec{0}\}$ is linearly dependent since, for example, $10\cdot\vec{0}=\vec{0}$.
2) Any set containing the vector $\vec{0}$ must be linearly dependent.
3) $S=\{\vec{v}\},\ \vec{v}\neq\vec{0}$, is linearly independent.
4) If $\vec{v}_1,\vec{v}_2,...,\vec{v}_k$ are vectors in a vector space V and any two of them are equal, then $\vec{v}_1,\vec{v}_2,...,\vec{v}_k$ are linearly dependent.

Geometric Interpretation of Linear Independence (see Fig 4.25 and 4.26 on p 224)

Theorem The nonzero vectors $\vec{v}_1,\vec{v}_2,...,\vec{v}_n$ in a vector space are linearly dependent if and only if one of the vectors $\vec{v}_j\ (j\geq 2)$ is a linear combination of the preceding vectors $\vec{v}_1,\vec{v}_2,...,\vec{v}_{j-1}$.

Proof:

("$\Leftarrow$") $\vec{v}_j=a_1\vec{v}_1+a_2\vec{v}_2+...+a_{j-1}\vec{v}_{j-1}$

$\Rightarrow a_1\vec{v}_1+a_2\vec{v}_2+...+a_{j-1}\vec{v}_{j-1}+(-1)\vec{v}_j+0\vec{v}_{j+1}+...+0\vec{v}_n=\vec{0}$

Since $-1\neq 0$, then $\vec{v}_1,\vec{v}_2,...,\vec{v}_n$ are linearly dependent.

("$\Rightarrow$") $\vec{v}_1,\vec{v}_2,...,\vec{v}_n$ linearly dependent

$\Rightarrow$ there exist scalars $a_1,a_2,...,a_n$ not all zero s.t.

$$a_1\vec{v}_1+a_2\vec{v}_2+...+a_n\vec{v}_n=\vec{0}.$$

Let j be the largest subscript for which $a_j\neq 0$. If $j\geq 2$, then

$$\vec{v}_j=-\left(\frac{a_1}{a_j}\right)\vec{v}_1-\left(\frac{a_2}{a_j}\right)\vec{v}_2-...-\left(\frac{a_{j-1}}{a_j}\right)\vec{v}_{j-1}.$$

If $j=1$, then

$a_1\vec{v}_1=\vec{0}\Rightarrow\vec{v}_1=\vec{0}$, a contradiction
since none of the vectors is the zero vector.

Thus, one of the vectors $\vec{v}_j$ is a linear combination of the preceding vectors $\vec{v}_1,\vec{v}_2,...,\vec{v}_{j-1}$.

Example $V=\mathbb{R}^3$

$$\vec{v}_1=\begin{bmatrix}1\\1\\-1\end{bmatrix},\ \vec{v}_2=\begin{bmatrix}-1\\2\\3\end{bmatrix},\ \vec{v}_3=\begin{bmatrix}-2\\1\\4\end{bmatrix},\ \vec{v}_4=\begin{bmatrix}3\\3\\3\end{bmatrix}$$

Note that $2\vec{v}_1+\vec{v}_2-\vec{v}_3-\vec{v}_4=\vec{0}$. This implies that $\vec{v}_1,\vec{v}_2,\vec{v}_3,\vec{v}_4$ are linearly dependent.
$\vec{v}_4=2\vec{v}_1+\vec{v}_2-\vec{v}_3$ is a linear combination of the preceding vectors.

Note also that

$$\vec{v}_2 = \vec{v}_1 + \vec{v}_3 + 0\vec{v}_4$$
$$\vec{v}_1 = \vec{v}_2 - \vec{v}_3 + 0\vec{v}_4$$
$$\vec{v}_3 = -\vec{v}_1 + \vec{v}_2 + 0\vec{v}_4$$

Remark $\vec{v}_1, \vec{v}_2, ..., \vec{v}_k$ linearly independent implies the vectors must be distinct and nonzero.

Remark Let $S = \{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}$ span a vector space *V* and let $\vec{v}_j$ be a linear combination of the preceding vectors in *S*. Then $S_1 = \{\vec{v}_1, \vec{v}_2, ..., \vec{v}_{j-1}, \vec{v}_{j+1}, ..., \vec{v}_n\} = S \setminus \{\vec{v}_j\}$ also spans *V*.

Proof:

Let $\vec{v} \in V$. Since *S* spans *V*, there exist scalars $a_1, a_2, ..., a_n$ s.t.

$$\vec{v} = a_1\vec{v}_1 + a_2\vec{v}_2 + ... + a_n\vec{v}_n.$$

If

$$\vec{v}_j = b_1\vec{v}_1 + b_2\vec{v}_2 + ... + b_{j-1}\vec{v}_{j-1},$$

then

$$\vec{v} = a_1\vec{v}_1 + a_2\vec{v}_2 + ... + a_j\left(b_1\vec{v}_1 + b_2\vec{v}_2 + ... + b_{j-1}\vec{v}_{j-1}\right) + a_{j+1}\vec{v}_{j+1} + ... + a_n\vec{v}_n$$
$$= c_1\vec{v}_1 + c_2\vec{v}_2 + ... + c_{j-1}\vec{v}_{j-1} + c_{j+1}\vec{v}_{j+1} + ... + c_n\vec{v}_n$$

Thus, $span\ S_1 = V$.

Example In the preceding example, if

$W = span\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$, then

$W = span\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ or $span\{\vec{v}_2, \vec{v}_3, \vec{v}_4\}$...

Exercises

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(a) Are $\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 4 & 6 \\ 8 & 6 \end{bmatrix}$ linearly independent?

Solution:

$a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 + a_4\vec{v}_4 = \vec{0}$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 4 & 0 \\ 1 & 0 & 3 & 6 & 0 \\ 2 & 0 & 2 & 8 & 0 \\ 1 & 2 & 1 & 6 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 4 & 0 \\ 0 & -1 & 3 & 2 & 0 \\ 0 & -2 & 2 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 3 & 6 & 0 \\ 0 & 1 & -3 & -2 & 0 \\ 0 & 0 & -4 & -4 & 0 \\ 0 & 0 & 4 & 4 & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 3 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{r} a_1 = -3r \\ a_2 = -r \\ a_3 = -r \\ a_4 = r,\ r \text{ real} \end{array} \Rightarrow \text{NOT linearly independent}$$

For ex, solution is $\begin{bmatrix} -3 \\ -1 \\ -1 \\ 1 \end{bmatrix}$.

(b) Are $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$ linearly independent?

Solution:

$a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 + a_4\vec{v}_4 = \vec{0}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 2 & 2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} a_1 = 0 \\ a_2 = 0 \Rightarrow \text{YES} \\ a_3 = 0 \end{array}$$

#14 a) $\cos t,\ \sin t,\ e^t$

Solution:

$a_1 \cos t + a_2 \sin t + a_3 e^t = 0 \Rightarrow a_1 = a_2 = a_3 = 0$

b) $t,\ e^t,\ \sin t$

Solution:

$a_1 t + a_2 e^t + a_3 \sin t = 0 \Rightarrow a_1 = a_2 = a_3 = 0$

c) $t^2,\ t,\ e^t$

Solution:

$$a_1 t^2 + a_2 t + a_3 e^t = 0 \Rightarrow a_1 = a_2 = a_3 = 0$$

d) $\cos^2 t,\ \sin^2 t,\ \cos 2t$

Solution:

Note that $\cos^2 t - \sin^2 t + \cos 2t = 0 \Rightarrow$ linearly dependent

#16 For what values of c are the vectors $[-1\ \ 0\ \ -1]$, $[2\ \ 1\ \ 2]$, $[1\ \ 1\ \ c]$ in $\mathbb{R}_3$ linearly dependent?

Solution:

$$\left[\begin{array}{ccc|c} -1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 2 & c & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & c-1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c-1 & 0 \end{array}\right]$$

linearly dependent if $c-1=0$ or $c=1$

#18 Let $\vec{u}$ and $\vec{v}$ be nonzero vectors in a vector space V.

Show: $\vec{u},\vec{v}$ linearly dependent if and only if there is a scalar k s.t. $\vec{v} = k\vec{u}$.

Proof:

$("\Rightarrow")$ $\vec{u},\vec{v}$ linearly dependent

$\Rightarrow \exists$ scalars a_1, a_2 not both zero s.t.

$$a_1\vec{u} + a_2\vec{v} = \vec{0}.$$

Suppose $a_2 \neq 0$, then $\vec{v} = -\dfrac{a_1}{a_2}\vec{u}$. Let $k = -\dfrac{a_1}{a_2}$.

$("\Leftarrow")$ Let $\vec{v} = k\vec{u}$ for some scalar k

$\Rightarrow -k\vec{u} + \vec{v} = \vec{0}$

$\Rightarrow -k\vec{u} + 1\cdot\vec{v} = \vec{0}$

$\Rightarrow$ linear dependence

#20 Let $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ be linearly independent. Prove that $T = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ is linearly independent where

$$\begin{aligned} \vec{w}_1 &= \vec{v}_1 + \vec{v}_2 + \vec{v}_3 \\ \vec{w}_2 &= \phantom{\vec{v}_1 + {}} \vec{v}_2 + \vec{v}_3 \\ \vec{w}_3 &= \phantom{\vec{v}_1 + \vec{v}_2 + {}} \vec{v}_3 \end{aligned}$$

Proof:

To show: $a_1\vec{w}_1 + a_2\vec{w}_2 + a_3\vec{w}_3 = \vec{0} \Rightarrow a_1 = a_2 = a_3 = 0.$

$$a_1\vec{w}_1 + a_2\vec{w}_2 + a_3\vec{w}_3 = \vec{0} \Rightarrow a_1\left(\vec{v}_1 + \vec{v}_2 + \vec{v}_3\right) + a_2\left(\vec{v}_2 + \vec{v}_3\right) + a_3\vec{v}_3 = \vec{0}$$

$$\Rightarrow a_1\vec{v}_1 + (a_1 + a_2)\vec{v}_2 + (a_1 + a_2 + a_3)\vec{v}_3 = \vec{0}$$

$$\Rightarrow \begin{aligned} a_1 &= 0 \\ a_1 + a_2 &= 0 \\ a_1 + a_2 + a_3 &= 0 \end{aligned} \quad \text{since } \vec{v}_1, \vec{v}_2, \vec{v}_3 \text{ are lin. ind.}$$

$$\Rightarrow a_1 = a_2 = a_3 = 0$$