Math 270 Linear Algebra

Chapter 4 Real Vector Spaces

4.4 Span

Definition If $S=\{\vec{v}_1,\vec{v}_2,...,\vec{v}_k\}$ is a set of vectors in a vector space V, then the set of all vectors in V that are linear combinations of the vectors in S is denoted by span S or span $\{\vec{v}_1,\vec{v}_2,...,\vec{v}_k\}$.

Geometric Interpretation in $\mathbb{R}^3$:
(Fig 4.22 on p 202)

span $\{\vec{v}_1,\vec{v}_2\}$ = plane that passes through the origin and contains $\vec{v}_1,\vec{v}_2$

Example 1 $S=\left\{\begin{bmatrix}1&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&1&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&1&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&1\end{bmatrix}\right\}\subseteq M_{23}$

To find span S:

Form $a\begin{bmatrix}1&0&0\\0&0&0\end{bmatrix}+b\begin{bmatrix}0&1&0\\0&0&0\end{bmatrix}+c\begin{bmatrix}0&0&0\\0&1&0\end{bmatrix}+d\begin{bmatrix}0&0&0\\0&0&1\end{bmatrix}$, a,b,c,d real

Thus, span S = subset of M_{23} consisting of matrices of the form $\begin{bmatrix}a&b&0\\0&c&d\end{bmatrix}$.

Theorem Let $S=\{\vec{v}_1,\vec{v}_2,...,\vec{v}_k\}$ be a set of vectors in a vector space V. Then span S is a subspace of V.

Proof: Let $\vec{u}$, $\vec{w}\in$ span S. Then

$$\vec{u}=\sum_{j=1}^{k}a_j\vec{v}_j \text{ and } \vec{w}=\sum_{j=1}^{k}b_j\vec{v}_j$$

for some real numbers $a_1,a_2,\ldots,a_k$ and $b_1,b_2,\ldots,b_k$.

$$\vec{u}+\vec{w}=\vec{u}=\sum_{j=1}^{k}a_j\vec{v}_j+\sum_{j=1}^{k}b_j\vec{v}_j=\sum_{j=1}^{k}\left(a_j+b_j\right)\vec{v}_j$$

$$\Rightarrow \vec{u}+\vec{w}\in \text{ span } S$$

For any real number c,

$$c\vec{u}=c\sum_{j=1}^{k}a_j\vec{v}_j=\sum_{j=1}^{k}\left(ca_j\right)\vec{v}_j \Rightarrow c\vec{u}\in \text{ span } S.$$

Thus, span S is a subspace of V.

Exercise #4 Determine whether the given vector A in M_{22} belongs to span $\{A_1, A_2, A_3\}$, where

$$A_1 = \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, A_3 = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}.$$

a) $A = \begin{bmatrix} 5 & 1 \\ -1 & 9 \end{bmatrix}$

b) $A = \begin{bmatrix} -3 & -1 \\ 3 & 2 \end{bmatrix}$

c) $A = \begin{bmatrix} 3 & -2 \\ 3 & 2 \end{bmatrix}$

d) $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

To find scalars a, b, c s.t. $A = aA_1 + bA_2 + cA_3$

$$\begin{bmatrix} 5 & 1 \\ -1 & 9 \end{bmatrix} = a\begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix} + b\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} + c\begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 1 \\ -1 & 9 \end{bmatrix} = \begin{bmatrix} a+b+2c & -a+b+2c \\ -c & 3a+2b+c \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 5 \\ -1 & 1 & 2 & 1 \\ 0 & 0 & -1 & -1 \\ 3 & 2 & 1 & 9 \end{array}\right]$$

Note that we need to do the same thing for (b), (c), and (d) so we can save time by doing all 4 at once as follows:

$$\left[\begin{array}{ccc|cccc} 1 & 1 & 2 & 5 & -3 & 3 & 1 \\ -1 & 1 & 2 & 1 & -1 & -2 & 0 \\ 0 & 0 & -1 & -1 & 3 & 3 & 2 \\ 3 & 2 & 1 & 9 & 2 & 2 & 1 \end{array}\right] \xrightarrow{\text{pivot on } a_{11}} \left[\begin{array}{ccc|cccc} 1 & 1 & 2 & 5 & -3 & 3 & 1 \\ 0 & 2 & 4 & 6 & -4 & 1 & 1 \\ 0 & 0 & -1 & -1 & 3 & 3 & 2 \\ 0 & -1 & -5 & -6 & 11 & -4 & -2 \end{array}\right]$$

$$\xrightarrow[\text{not crucial}]{-R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|cccc} 1 & 0 & 2 & 5 & -3 & 3 & 1 \\ 0 & 1 & 5 & 6 & 11 & 4 & 2 \\ 0 & 0 & -1 & -1 & 3 & 3 & 2 \\ 0 & 2 & 4 & 6 & -4 & 1 & 1 \end{array}\right] \xrightarrow{\text{pivot on } a_{22}} \left[\begin{array}{ccc|cccc} 1 & 0 & -3 & -1 & 8 & -4 & -1 \\ 0 & 1 & 5 & 6 & -11 & 7 & 2 \\ 0 & 0 & -1 & -1 & 3 & 3 & 2 \\ 0 & 0 & -6 & -6 & 18 & -13 & -3 \end{array}\right]$$

$$\xrightarrow{\text{pivot on } a_{33}} \left[\begin{array}{ccc|cccc} 1 & 0 & 0 & 2 & -1 & -13 & -7 \\ 0 & 1 & 0 & 1 & 4 & 22 & 12 \\ 0 & 0 & 1 & 1 & -3 & -3 & -2 \\ 0 & 0 & 0 & 0 & 0 & -31 & -15 \end{array}\right]$$

yes yes no no

Definition The vectors $\vec{v}_1, \vec{v}_2, ..., \vec{v}_k$ in a vector space V are said to **<u>span</u>** V if every vector in V is a linear combination of $\vec{v}_1, \vec{v}_2, ..., \vec{v}_k$. If $S = \{\vec{v}_1, \vec{v}_2, ..., \vec{v}_k\}$, then we say S **<u>spans</u>** V, or $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_k\}$ **<u>spans</u>** V, or V is **<u>spanned by</u>** S or span $S = V$.

Exercise #8 Which span P_2 ?

a) $\{t^2+1,\ t^2+t,\ t+1\}$

Solution:

Take $at^2+bt+c \in P_2$. To determine whether there are constants a_1, a_2, a_3 s.t.

$$a_1(t^2+1)+a_2(t^2+t)+a_3(t+1)=at^2+bt+c.$$

This means solve the linear system

$$\begin{aligned} a_1 + a_2 &= a \\ a_2 + a_3 &= b \\ a_1 + a_3 &= c \end{aligned}$$

Or

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 0 & 1 & 1 & b \\ 1 & 0 & 1 & c \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 0 & 1 & 1 & b \\ 0 & -1 & 1 & c-a \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & a-b \\ 0 & 1 & 1 & b \\ 0 & 0 & 2 & c-a+b \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & a-b+\frac{c-a+b}{2} \\ 0 & 1 & 0 & b-\frac{c-a+b}{2} \\ 0 & 0 & 1 & \frac{c-a+b}{2} \end{array}\right] \Rightarrow \text{YES}$$

b) $\{t^2+1,\ t-1,\ t^2+t\}$

Solution:

Take $at^2+bt+c \in P_2$. To determine whether there are constants a_1, a_2, a_3 s.t.

$$a_1(t^2+1)+a_2(t-1)+a_3(t^2+t)=at^2+bt+c.$$

Solve:

$$\begin{aligned} a_1 + a_3 &= a \\ a_2 + a_3 &= b \\ a_1 - a_2 &= c \end{aligned}$$

$$\begin{bmatrix} 1 & 0 & 1 & | & a \\ 0 & 1 & 1 & | & b \\ 1 & -1 & 0 & | & c \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 & | & a \\ 0 & 1 & 1 & | & b \\ 0 & -1 & -1 & | & c-a \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 & | & a \\ 0 & 1 & 1 & | & b \\ 0 & 0 & 0 & | & c-a+b \end{bmatrix} \Rightarrow \text{NO}$$

Exercise #10 Does the set $S = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right\}$ span M_{22} ?

Solution:

Take $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{22}$. To determine whether there are constants a_1, a_2, a_3, a_4 s.t.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a_1 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + a_3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + a_4 \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} a &= a_1 + a_3 \\ b &= a_1 + a_4 \\ c &= a_2 + a_4 \\ d &= a_2 + a_3 + a_4 \end{aligned} \Rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 & | & a \\ 1 & 0 & 0 & 1 & | & b \\ 0 & 1 & 0 & 1 & | & c \\ 0 & 1 & 1 & 1 & | & d \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & 1 & 0 & | & a \\ 0 & 0 & -1 & 1 & | & b-a \\ 0 & 1 & 0 & 1 & | & c \\ 0 & 1 & 1 & 1 & | & d \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 & 0 & | & a \\ 0 & 0 & -1 & 1 & | & b-a \\ 0 & 1 & 0 & 1 & | & c \\ 0 & 0 & 1 & 0 & | & d-c \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & | & a-d+c \\ 0 & 0 & 0 & 1 & | & b-a+d-c \\ 0 & 1 & 0 & 1 & | & c \\ 0 & 0 & 1 & 0 & | & d-c \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & | & a-d+c \\ 0 & 0 & 0 & 1 & | & b-a+d-c \\ 0 & 1 & 0 & 0 & | & c \\ 0 & 0 & 1 & 0 & | & d-c \end{bmatrix} \Rightarrow \text{YES}$$

Note: I did not double-check this…but the answer is YES.