Math 270 Linear Algebra

Chapter 4 Real Vector Spaces

4.3 Subspaces

Definition Let $\langle V,\oplus,\odot\rangle$ be a vector space and let $W \subseteq V$. If $\langle W,\oplus,\odot\rangle$ is a vector space then W is called a **<u>subspace</u>** of V.

Example 1 The **<u>trivial</u>** subspaces of V are itself and the **<u>zero subspace</u>** $\{\vec{0}\}$.

Example 2 $P_2 =$ set of all polynomials of degree ≤ 2 and the zero polynomial
$P_2 \subseteq P$, where P is the vector space of all polynomials
P_2 is a subspace of P

Remarks

1. In general, P_n is a subspace of P, where P_n is the set of all polynomials of degree $\leq n$.
2. P_n is a subspace of P_{n+1}.

Example 3 $V =$ set of polynomials of degree *exactly* 2
$V \subset P$ but V is <u>not</u> a subspace of P, since for example, $t^2+4 \in V$ and $-t^2+1 \in V$, but the sum is degree 0, which is <u>not</u> in V

How to verify if W is a subspace of V without checking all 10 conditions:

Theorem Let $\langle V,\oplus,\odot\rangle$ be a vector space and $W \subseteq V$, $W \neq \varnothing$.
W is a subspace of V if and only if (a) and (b) are satisfied:

(a) $\vec{u},\vec{v} \in W \Rightarrow \vec{u}\oplus\vec{v} \in W$
(b) c real, $\vec{u} \in W \Rightarrow c\odot\vec{u} \in W$

Proof:

$("\Rightarrow")$ W subspace of $V \Rightarrow (a),(b),(1)-(8)$ hold $\Rightarrow (a),(b)$ hold

$("\Leftarrow")$ Let (a) and (b) hold. To show (1) – (8) also hold.

$(b) \Rightarrow (-1)\odot\vec{u} \in W$ (4)

$(a) \Rightarrow \vec{u}\oplus(-1)\odot\vec{u} \in W$

$\Rightarrow \vec{0} \in$ W

$\Rightarrow \vec{u}\oplus\vec{0} = \vec{u}$ for any $\vec{u} \in W$ (3)

(1), (2), (5), (6), (7), (8) follow since $\vec{u},\vec{v} \in W \Rightarrow \vec{u},\vec{v} \in V$

Example 4

a) W_1 = set of all vectors of the form $\begin{bmatrix} x \\ y \end{bmatrix}$, where $x \geq 0$

W_1 is <u>not</u> a subspace of $\mathbb{R}^2$ since if we consider the scalar $c = -1$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \in W_1$, then

$(-1) \odot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} \notin W_1$

b) W_2 = set of all vectors of the form $\begin{bmatrix} x \\ y \end{bmatrix}$, where $x \geq 0,\ y \geq 0$

W_2 is <u>not</u> a subspace of $\mathbb{R}^2$, since if $c = -1$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \in W_2$, then $(-1) \odot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} \notin W_2$.

c) W_3 = set of all vectors of the form $\begin{bmatrix} x \\ y \end{bmatrix}$ where $x = 0$

W_3 is a subspace of $\mathbb{R}^2$ since:

$$\begin{bmatrix} 0 \\ b_1 \end{bmatrix}, \begin{bmatrix} 0 \\ b_2 \end{bmatrix} \in W_3 \Rightarrow \begin{bmatrix} 0 \\ b_1 \end{bmatrix} \oplus \begin{bmatrix} 0 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ b_1 + b_2 \end{bmatrix} \in W_3$$

$$c \odot \begin{bmatrix} 0 \\ b_1 \end{bmatrix} = \begin{bmatrix} 0 \\ cb_1 \end{bmatrix} \in W_3$$

Example 5 W = set of all vectors in $\mathbb{R}^3$ of the form $\begin{bmatrix} a \\ b \\ a+b \end{bmatrix}$ where $a, b \in \mathbb{R}$

W is a subspace of $\mathbb{R}^3$.

Notation Change:

$\oplus = +$ $\qquad \vec{u} \oplus \vec{v} = \vec{u} + \vec{v}$

$\odot = \cdot$ $\qquad c \odot \vec{u} = c\vec{u}$

Remark To show that $\varnothing \neq W \subseteq V$ is a subspace of a vector space V, we can show that $c\vec{u} + d\vec{v} \in W$ for any vectors $\vec{u}, \vec{v} \in W$ and for any scalars c and d, i.e.

$$W \text{ is a subspace of } V \Leftrightarrow c\vec{u} + d\vec{v} \in W$$

This will take care of (a) and (b), or the closure properties.

Exercise #6a W = set of all vectors of the form $\begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$; $W \subseteq \mathbb{R}^3$

W_3 is a subspace of $\mathbb{R}^3$ since for any scalars r and s,

$$r\begin{bmatrix} a_1 \\ b_1 \\ 0 \end{bmatrix} + s\begin{bmatrix} a_2 \\ b_2 \\ 0 \end{bmatrix} = \begin{bmatrix} ra_1 + sa_2 \\ rb_1 + sb_2 \\ 0 \end{bmatrix} \in W.$$

Exercise #6b W = set of all vectors of the form $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, where $a > 0$

W is not a subspace of $\mathbb{R}^3$ since $(-1)\begin{bmatrix} a \\ b \\ c \end{bmatrix} \notin W$.

Exercise #6c W = set of all vectors of the form $\begin{bmatrix} a \\ a \\ c \end{bmatrix}$.

W is a subspace of $\mathbb{R}^3$ since

$$r\begin{bmatrix} a_1 \\ a_1 \\ c_1 \end{bmatrix} + s\begin{bmatrix} a_2 \\ a_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} ra_1 + sa_2 \\ ra_1 + sa_2 \\ c_1 + c_2 \end{bmatrix} \in W$$

Exercise #6d W = set of all vectors of the form $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, where $2a - b + c = 1$.

$$r\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} + s\begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} ra_1 + sa_2 \\ rb_1 + sb_2 \\ rc_1 + sc_2 \end{bmatrix}$$

$$2(ra_1 + sa_2) - (rb_1 + sb_2) + (rc_1 + sc_2)$$
$$= r(2a_1 - b_1 + c_1) + s(2a_2 - b_2 + c_2)$$
$$= r(1) + s(1)$$
$$= r + s \neq 1$$

W is not a subspace of $\mathbb{R}^3$.

Note: We could have shown #6d by considering condition (b). If we take the scalar $r = 0$, then for $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \in W$ then $r\begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and $2(0) - 0 + 0 = 0 \neq 1$

How to construct a subspace in a vector space:

Let $\vec{v}_1, \vec{v}_2 \in V$, V vector space .

Let W = set of all vectors in V of the form $a_1\vec{v}_1 + a_2\vec{v}_2,\ a_1, a_2 \in \mathbb{R}$, i.e., all vectors in V that are **<u>linear combinations</u>** of $\vec{v}_1$ and $\vec{v}_2$.

W is a subspace of V since:

(a) $\vec{w}_1 = a_1\vec{v}_1 + a_2\vec{v}_2$ and $\vec{w}_2 = b_1\vec{v}_1 + b_2\vec{v}_2$

$$\Rightarrow \vec{w}_1 + \vec{w}_2 = \left(a_1\vec{v}_1 + a_2\vec{v}_2\right) + \left(b_1\vec{v}_1 + b_2\vec{v}_2\right)$$

$$= \left(a_1 + b_1\right)\vec{v}_1 + \left(a_2 + b_2\right)\vec{v}_2 \in W$$

(b) c scalar,

$$c\vec{w}_1 = c\left(a_1\vec{v}_1 + a_2\vec{v}_2\right) = \left(ca_1\right)\vec{v}_1 + \left(ca_2\right)\vec{v}_2 \in W$$

Note: The construction can be extended to more than 2 vectors.

Geometric Interpretation: (see Fig. 4.20 on p. 202)

Exercise #34a Is $\begin{bmatrix}3 & 6 & 3 & 0\end{bmatrix} \in \mathbb{R}_4$ a linear combination of

$$\vec{v}_1 = \begin{bmatrix}1 & 2 & 1 & 0\end{bmatrix}, \vec{v}_2 = \begin{bmatrix}4 & 1 & -2 & 3\end{bmatrix}, \vec{v}_3 = \begin{bmatrix}1 & 2 & 6 & -5\end{bmatrix}, \vec{v}_4 = \begin{bmatrix}0 & 0 & 0 & 1\end{bmatrix} \ ?$$

Solution:

Can we find scalars a_1, a_2, a_3, a_4 s.t.

$$\begin{bmatrix}3 & 6 & 3 & 0\end{bmatrix} = a_1\begin{bmatrix}1 & 2 & 1 & 0\end{bmatrix} + a_2\begin{bmatrix}4 & 1 & -2 & 3\end{bmatrix} + a_3\begin{bmatrix}1 & 2 & 6 & -5\end{bmatrix} + a_4\begin{bmatrix}0 & 0 & 0 & 1\end{bmatrix} \ ?$$

$$\left[\begin{array}{cccc|c}1 & 4 & 1 & 0 & 3\\ 2 & 1 & 2 & 0 & 6\\ 1 & -2 & 6 & 0 & 3\\ 0 & 3 & -5 & 1 & 0\end{array}\right] \xrightarrow{\text{pivot on } a_{11}} \left[\begin{array}{cccc|c}1 & 4 & 1 & 0 & 3\\ 0 & -7 & 0 & 0 & 0\\ 0 & -6 & 5 & 0 & 0\\ 0 & 3 & -5 & 1 & 0\end{array}\right] \xrightarrow{-R_3 + R_2} \left[\begin{array}{cccc|c}1 & 4 & 1 & 0 & 3\\ 0 & -1 & -5 & 0 & 0\\ 0 & -6 & 5 & 0 & 0\\ 0 & 3 & -5 & 1 & 0\end{array}\right]$$

$$\xrightarrow{\text{pivot on } a_{22}} \left[\begin{array}{cccc|c}1 & 0 & -19 & 0 & 3\\ 0 & 1 & 5 & 0 & 0\\ 0 & 0 & 35 & 0 & 0\\ 0 & 0 & -20 & 1 & 0\end{array}\right] \xrightarrow{\text{pivot on } a_{33}} \left[\begin{array}{cccc|c}1 & 0 & 0 & 0 & 3\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\end{array}\right] \Rightarrow a_1 = 3, a_2 = a_3 = a_4 = 0$$

Note: We could have gotten the answer by inspection, but the above method is the general method.

Geometric Interpretation of the Set of Linear Combinations of Two Vectors in $\mathbb{R}^3$:
(see Fig. 4.22 on p. 202)

Example 6 A m x n

$A\vec{x} = \vec{0},\ \ \vec{x} \in \mathbb{R}^n$

$W =$ set of all solutions

$W \subset \mathbb{R}^n$

To show: W is a subspace of $\mathbb{R}^n$

Solution:

Let $\vec{x}$ and $\vec{y}$ be solutions. Then $A\vec{x}=\vec{0}$ and $A\vec{y}=\vec{0}$.

To show $\vec{x}+\vec{y}$ is a solution.

$$A(\vec{x}+\vec{y})=A\vec{x}+A\vec{y}=\vec{0}+\vec{0}=\vec{0}$$

Thus, $\vec{x}+\vec{y}\in W$.

To show $c\vec{x}$ is a solution for scalar c:

$$A(c\vec{x})=cA\vec{x}=c\vec{0}=\vec{0}$$

Thus, $c\vec{x}\in W$.

Remarks

1. W is called the **<u>solution space</u>** of $A\vec{x}=\vec{0}$ or the **<u>null space</u>** of A.
2. The set of all solutions to $A\vec{x}=\vec{b}$ is <u>not</u> a subspace of $\mathbb{R}^n$ (Exercise 23).
3. The subspaces of $\mathbb{R}^1$ are $\{0\}$ and $\mathbb{R}^1$ and the subspaces of $\mathbb{R}^2$ are $\{0\}$, $\mathbb{R}^2$, and any set consisting of all scalar multiples of a nonzero vector