Math 270 Linear Algebra

Chapter 4 Real Vector Spaces

4.1 Vectors in the Plane and in 3-Space

2D: $\vec{u} = \begin{bmatrix} x \\ y \end{bmatrix}$; 3D: $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Properties If $\vec{u}$, $\vec{v}$, $\vec{w}$ are vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$, and *c*, *d* are real scalars, then

(a) $\vec{u}+\vec{v}=\vec{v}+\vec{u}$

(b) $\left(\vec{u}+\vec{v}\right)+\vec{w}=\vec{u}+\left(\vec{v}+\vec{w}\right)$

(c) $\vec{u}+\vec{0}=\vec{0}+\vec{u}=\vec{u}$

(d) $\vec{u}+\left(-\vec{u}\right)=\vec{0}$

(e) $c\left(\vec{u}+\vec{v}\right)=c\vec{u}+c\vec{v}$

(f) $(c+d)\vec{u}=c\vec{u}+d\vec{u}$

(g) $c\left(d\vec{u}\right)=(cd)\vec{u}$

(h) $1\,\vec{u}=\vec{u}$

4.2 Vector Spaces

Definition A **<u>real vector space</u>** is a set *V* of elements on which we have two operations $\oplus$ and $\odot$, $\left(\langle V,\oplus,\odot\rangle\right)$ defined with the following properties:

(a) $\vec{u},\vec{v}\in V \Rightarrow \vec{u}\oplus\vec{v}\in V$ (*V* is **<u>closed</u>** under $\oplus$.)

- **1)** $\vec{u}\oplus\vec{v}=\vec{v}\oplus\vec{u}, \ \forall \vec{u},\vec{v}\in V$
- **2)** $\vec{u}\oplus\left(\vec{v}\oplus\vec{w}\right)=\left(\vec{u}\oplus\vec{v}\right)\oplus\vec{w}, \ \forall \vec{u},\vec{v},\vec{w}\in V$
- **3)** There is an element $\vec{0}$ in *V* s.t. $\vec{u}\oplus\vec{0}=\vec{0}\oplus\vec{u}=\vec{u}$ for any $\vec{u}\in V$.
- **4)** For each $\vec{u}\in V$, there is an element $-\vec{u}\in V$ s.t. $\vec{u}\oplus\left(-\vec{u}\right)=\left(-\vec{u}\right)\oplus\vec{u}=\vec{0}$

(b) $\vec{u}\in V, c\in\mathbb{R} \Rightarrow c\odot\vec{u}\in V$ (*V* is closed under $\odot$.)

- **1)** $c\odot\left(\vec{u}\oplus\vec{v}\right)=c\odot\vec{u}\oplus c\odot\vec{v}$, for any $\vec{u},\vec{v}\in V$, c real
- **2)** $(c+d)\odot\vec{u}=c\odot\vec{u}+d\odot\vec{u}$, for any $\vec{u}\in V$ and any c,d real
- **3)** $c\odot\left(d\odot\vec{u}\right)=(cd)\odot\vec{u}$ for any $\vec{u}\in V$ and any real c,d
- **4)** $1\odot\vec{u}=\vec{u}$ for any $\vec{u}\in V$

The elements of *V* are called **<u>vectors</u>**; the elements of $\mathbb{R}$ are called **<u>scalars</u>**; $\oplus$ is called **<u>vector addition</u>**; $\odot$ is called **<u>scalar multiplication</u>**. $\vec{0}$ is called the **<u>zero vector</u>**; $-\vec{u}$ is called the **<u>negative</u>** of $\vec{u}$. $\vec{0}$ and $-\vec{u}$ are both unique. (Exercises 19 and 20).

Note: If we allow the scalars to be complex, we get a **<u>complex vector space</u>**. We will focus on real vector spaces.

Example 1 $\mathbb{R}^n$ = set of all $n \times 1$ matrices $\begin{bmatrix} a_1 \\ a_2 \\ \\ a_n \end{bmatrix}$

$\oplus$ = matrix addition

$\odot$ = scalar multiplication

$\langle \mathbb{R}^n, \oplus, \odot \rangle$ is a vector space

Example 2 M_{mn} = set of all m x n matrices

$\oplus$ = matrix addition

$\odot$ = scalar multiplication

$\langle M_{mn}, \oplus, \odot \rangle$ is a vector space

Example 3 $\mathbb{R}$ = set of real numbers

$\oplus = +$, usual addition of reals

$\odot = \cdot$, usual multiplication of reals

$\langle \mathbb{R}, \oplus, \odot \rangle = \langle \mathbb{R}, +, \cdot \rangle$ is a vector space

Example 4 $\mathbb{R}_n$ = set of all 1 x n matrices $[a_1 \quad a_2 \quad \ldots \quad a_n]$

$\oplus$: $[a_1 \quad a_2 \quad \ldots \quad a_n] \oplus [b_1 \quad b_2 \quad \ldots \quad b_n] = [a_1 + b_1 \quad a_2 + b_2 \quad \ldots \quad a_n + b_n]$

$\odot$: $c \odot [a_1 \quad a_2 \quad \ldots \quad a_n] = [ca_1 \quad ca_2 \quad \ldots \quad ca_n]$

$\langle \mathbb{R}_n, \oplus, \odot \rangle$ is a vector space

Example 5 <u>**polynomial**</u> $p(t) = a_n t^n + a_{n-1} t^{n-1} + \ldots + a_1 t + a_0$, where $a_0, a_1, \ldots, a_n$ are real numbers, and n is a nonnegative integer; if $a_n \neq 0$, n is the <u>**degree**</u> of $p(t)$; <u>**zero polynomial**</u>= $\vec{0}$

Let P_n = set of all polynomials of degree $\leq n$ together with the zero polynomial

For $p(t)$ and $g(t)$ in P_n,

$$p(t) = a_n t^n + a_{n-1} t^{n-1} + \ldots + a_1 t + a_0$$
$$g(t) = b_n t^n + b_{n-1} t^{n-1} + \ldots + b_1 t + b_0$$

define

$$p(t) \oplus g(t) = (a_n + b_n) t^n + (a_{n-1} + b_{n-1}) t^{n-1} + \ldots + (a_1 + b_1) t + (a_0 + b_0).$$

For scalar c, define

$$c \odot p(t) = (ca_n) t^n + (ca_{n-1}) t^{n-1} + \ldots + (ca_1) t + (ca_0)$$

$\langle P_n, \oplus, \odot \rangle$ is a vector space.

Example 6 V = set of all real-valued continuous functions defined on $\mathbb{R}$; $f \in V$, $f : \mathbb{R} \to \mathbb{R}$.

Define for $f, g \in V$:

$$f \oplus g = (f \oplus g)(t) = f(t) + g(t).$$

If c is a scalar,

$$c \odot f = (c \odot f)(t) = cf(t).$$

Then $\langle V, \oplus, \odot \rangle$ is a vector space.

Notation: $C(-\infty, \infty)$

Example 7 $V = \mathbb{R}$

$\oplus =$ ordinary subtraction, i.e. for $\vec{u}, \vec{v} \in V$, $\vec{u} \oplus \vec{v} = \vec{u} - \vec{v}$

$\odot =$ ordinary multiplication, i.e. for c scalar, $c \odot \vec{u} = c\vec{u}$

$\langle V, \oplus, \odot \rangle$ is <u>NOT</u> a vector space since $\vec{u} \oplus \vec{v} \neq \vec{v} \oplus \vec{u}$.

Example 8 $V =$ set of all ordered triples of real numbers (x, y, z)

$\oplus$: $(x, y, z) \oplus (x', y', z') = (x', y + y', z + z')$

$\odot$: $c \odot (x, y, z) = (cx, cy, cz)$

$\langle V, \oplus, \odot \rangle$ is <u>NOT</u> a vector space since for $\vec{u} = (x, y, z)$ and $\vec{v} = (x', y', z')$,

$$\vec{u} \oplus \vec{v} = (x, y, z) \oplus (x', y', z') = (x, y + y', z + z')$$

$$\vec{v} \oplus \vec{u} = (x', y', z') \oplus (x, y, z) = (x', y' + y, z' + z)$$

are not equal.

Example 9 $V =$ set of all integers

$\oplus =$ ordinary addition

$\odot =$ ordinary multiplication

$\langle V, \oplus, \odot \rangle$ is <u>NOT</u> a vector space since for an irrational scalar c, say $\sqrt{2}$,

$$c \odot \vec{u} = \sqrt{2}\vec{u} \notin V.$$

Some useful properties common to all vector spaces:

Theorem If $\langle V, \oplus, \odot \rangle$ is a vector space, then

(a) $0 \odot \vec{u} = \vec{0}$ for any $\bar{u} \in V$

(b) $c \odot \vec{0} = \vec{0}$ for any scalar c

(c) $c \odot \vec{u} = \vec{0} \Rightarrow c = 0$ or $\vec{u} = \vec{0}$

(d) $(-1) \odot \vec{u} = -\ \vec{u}$ for any $\vec{u} \in V$

Proof:

(a)

$$0 \odot \vec{u} = (0 + 0) \oplus \vec{u}$$

$$0 \odot \vec{u} = 0 \odot \vec{u} + 0 \odot \vec{u}$$

$$-0 \odot \vec{u} + 0 \odot \vec{u} = -0 \odot \vec{u} + \left(0 \odot \vec{u} + 0 \odot \vec{u}\right)$$

$$\vec{0} = 0 \odot \vec{u}$$

(b)

$$c \odot \vec{0} = c \odot \left(\vec{0} \oplus \vec{0}\right) = c \odot \vec{0} \oplus c \odot \vec{0}$$

$$\Rightarrow c \odot \vec{0} = \vec{0}$$

Exercises

#14 $\langle V,\oplus,\odot\rangle = \langle\{0\},\oplus,\odot\rangle$

where $\vec{0}\oplus\vec{0}=\vec{0}$

$c\odot\vec{0}=\vec{0}$

Prove V is a vector space.

Proof:

a) $\vec{0}\oplus\vec{0}=\vec{0}\in V$

1) $\vec{0}\oplus\vec{0}=\vec{0}\oplus\vec{0}=\vec{0}$

2) $\vec{0}+\left(\vec{0}\oplus\vec{0}\right)=\left(\vec{0}\oplus\vec{0}\right)\oplus\vec{0}$

3) $\vec{0}\in V$ s.t. $\vec{0}\oplus\vec{0}=\vec{0}$

4) $-\vec{u}=\vec{0}\in V$ s.t. $\vec{0}\oplus\vec{0}=\vec{0}$

b) c scalar, $c\odot\vec{0}=\vec{0}\in V$

1) $c\odot\left(\vec{0}\oplus\vec{0}\right)=c\odot\vec{0}=\vec{0}$; $c\odot\vec{0}\oplus c\odot\vec{0}=\vec{0}\oplus\vec{0}=\vec{0}$

2) $(c+d)\odot\vec{0}=\vec{0}$; $\left(c\odot\vec{0}\right)\oplus\left(d\odot\vec{0}\right)=\vec{0}+\vec{0}=\vec{0}$

3) $c\odot\left(d\odot\vec{0}\right)=c\odot\vec{0}=\vec{0}$; $(cd)\odot\vec{0}=\vec{0}$

4) $1\odot\vec{0}=\vec{0}$

#8 V = set of all ordered pairs of real numbers = $\{(x,y)\,|\,x,y\in\mathbb{R}\}$

$\oplus$: $(x,y)\oplus(x',y')=(x+x',y+y')$

$\odot$: $r\odot(x,y)=(x,ry)$

Show: $\langle V,\oplus,\odot\rangle$ is <u>NOT</u> a vector space.

Solution:

(5) Let $\vec{u}=(x,y)$, $\vec{v}=(x',y')$, $c\in\mathbb{R}$

$$\begin{aligned} c\odot\left(\vec{u}\oplus\vec{v}\right) &= c\odot(x+x',y+y') \\ &= \left(x+x',c(y+y')\right) \\ &= (x,cy)+(x',cy') \\ &= c\odot\vec{u}\oplus c\odot\vec{v} \end{aligned}$$

(6) $(c+d)\odot\vec{u}=\left(x,(c+d)y\right)=(x,cy+cy)$

$c\odot\vec{u}\oplus d\odot\vec{u}=(x,cy)+(x,dy)=(2x,cy+dy)$

#16 V = set of positive real numbers

$\oplus$: $\vec{u}\oplus\vec{v}=\vec{u}\vec{v}-1$

$\odot$: $c\odot\vec{v}=\vec{v}$

Is $\langle V,\oplus,\odot\rangle$ a vector space?

Solution:

$\langle V,\oplus,\odot\rangle$ is <u>NOT</u> a vector space since if $\vec{u}=\vec{v}=1$, $\vec{u}\oplus\vec{v}=(1)(1)-1=0\notin V$.

#18 V = set of all real numbers

$\oplus$: $\vec{u}+\vec{v}=2\vec{u}-\vec{v}$

$\odot$: $c\odot\vec{u}=c\vec{u}$

Is $\langle V,\oplus,\odot\rangle$ a vector space?

Solution:

$\langle V,\oplus,\odot\rangle$ is <u>NOT</u> a vector space since $\vec{u}\oplus\vec{v}=2\vec{u}-\vec{v}$; $\vec{v}\oplus\vec{u}=2\vec{v}-\vec{u}$.

#12 V = set of all positive real numbers = $\mathbb{R}^+$

$\oplus$: $\vec{u}\oplus\vec{v}=\vec{u}\vec{v}$ ordinary multiplication

$\odot$: $c\odot\vec{v}=\vec{v}^{\,c}$

Prove that V is a vector space.

Proof:

(a) Let $x,y\in\mathbb{R}^+$. Then $x\oplus y=xy\in\mathbb{R}^+$

(b) Let $x\in\mathbb{R}^+$ and c a real scalar. Then $c\odot x=x^c\in\mathbb{R}^+$

(1) $x\oplus y=xy=yx=y\oplus x$

(2) $x\oplus(y\oplus z)=x\oplus yz=x(yz)=(xy)z=(x\oplus y)\oplus z$

(3) $\vec{0}=1\in\mathbb{R}^+$ s.t. $x+\vec{0}=x\cdot\vec{0}=x\cdot 1=x$ for any $x\in\mathbb{R}^+$

(4) For each $x\in\mathbb{R}^+$, there is $-x=\dfrac{1}{x}$ s.t. $x\oplus -x=x\cdot\dfrac{1}{x}=1=\vec{0}$.

(5) $c\odot(x\oplus y)=c\odot xy=(xy)^c=x^cy^c=c\odot x+c\odot y$

(6) $(c+d)\odot x=x^{c+d}=x^c\cdot x^d=x^c\oplus x^d=c\odot x+d\odot x$

(7) $c\odot(d\odot x)=c\odot x^d=\left(x^d\right)^c=x^{cd}=(cd)\odot x$

(8) $1\odot x=x^1=x$ for any $x\in\mathbb{R}^+$