Math 270 Linear Algebra

Chapter 3 Determinants

3.4 Inverse of a Matrix

Theorem If $A = \left[a_{ij} \right]$ is n x n, then

$$a_{i1}A_{k1} + a_{i2}A_{k2} + \cdots + a_{in}A_{kn} = 0 \text{ for } i \neq k$$
$$a_{1j}A_{1k} + a_{2j}A_{2k} + \cdots + a_{nj}A_{nk} = 0 \text{ for } j \neq k$$

i.e., the sum of the products of the elements of any row (column) times the corresponding cofactors of any other row(column) is zero.

Example

$$A = \begin{bmatrix} 2 & 1 & -3 \\ 5 & 4 & 1 \\ 2 & 6 & 7 \end{bmatrix}$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} 1 & -3 \\ 6 & 7 \end{vmatrix} = (-1)(7+18) = -25$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 2 & -3 \\ 2 & 7 \end{vmatrix} = (1)(14+6) = 20$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 2 & 1 \\ 2 & 6 \end{vmatrix} = (-1)(12-2) = -10$$

$$a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23} = 2(-25) + 1(20) + (-3)(-10) = -50 + 20 + 30 = 0$$
$$a_{31}A_{21} + a_{32}A_{22} + a_{33}A_{23} = 2(-25) + 6(20) + 7(-10) = -50 + 120 + 70 = 0$$

Thus,

$$a_{i1}A_{k1} + a_{i2}A_{k2} + \cdots + a_{in}A_{kn} = \begin{cases} \det(A), & \text{if } i = k \\ 0, & \text{if } i \neq k \end{cases}$$

and

$$a_{j1}A_{1k} + a_{j2}A_{2k} + \cdots + a_{nj}A_{nk} = \begin{cases} \det(A), & \text{if } j = k \\ 0, & \text{if } j \neq k \end{cases}$$

Definition Let $A = \left[a_{ij} \right]$ be n x n. The n x n matrix adj A, called the **<u>adjoint</u>** of A, is the matrix whose (i, j)th entry is the cofactor A_{ji} of a_{ji}. Thus,

$$adj\ A = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}.$$

Example $A = \begin{bmatrix} 2 & 1 & -3 \\ 5 & 4 & 1 \\ 2 & 6 & 7 \end{bmatrix}$

$$A_{11} = (-1)^{1+1}\begin{vmatrix} 4 & 1 \\ 6 & 7 \end{vmatrix} = 28 - 6 = 22$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} 5 & 1 \\ 2 & 7 \end{vmatrix} = (-1)(35 - 2) = -33$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 5 & 4 \\ 2 & 6 \end{vmatrix} = 30 - 8 = 22$$

$$A_{21} = -25$$
$$A_{22} = 20$$
$$A_{23} = -10$$
$$A_{31} = 13$$
$$A_{32} = -17$$
$$A_{33} = 3$$

$$Adj\ A = \begin{bmatrix} 22 & -25 & 13 \\ -33 & 20 & -17 \\ 22 & -10 & 3 \end{bmatrix}$$

Theorem If $A = [a_{ij}]$ is n x n, then $A(adj\ A) = (adj\ A)A = \det(A)I_n$.

Proof:

$$A(adj\ A) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{j1} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{j2} & \cdots & A_{n2} \\ \vdots & \vdots & & \vdots & & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{jn} & \cdots & A_{nn} \end{bmatrix}$$

(i, j) th entry in the product is

$$a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn} = \begin{cases} \det(A), \text{ if } i = j \\ 0, \text{ if } i \neq j \end{cases}$$

$$\Rightarrow A(adj\ A) = \begin{bmatrix} \det(A) & & & \\ 0 & \det(A) & 0 & \\ 0 & 0 & \ddots & \\ & & 0 & \det(A) \end{bmatrix} = \det(A)I_n$$

Similarly, $(adj\ A)A = \det(A)I_n$.

Example

$$\begin{bmatrix} 2 & 1 & -3 \\ 5 & 4 & 1 \\ 2 & 6 & 7 \end{bmatrix}\begin{bmatrix} 22 & -25 & 13 \\ -33 & 20 & -17 \\ 22 & -10 & 3 \end{bmatrix} = \begin{bmatrix} -55 & 0 & 0 \\ 0 & -55 & 0 \\ 0 & 0 & -55 \end{bmatrix} = -55I_3$$

Corollary If A is n x n and $\det(A) \neq 0$, then $A^{-1} = \dfrac{1}{\det(A)}(adj\ A)$.

Example $A = \begin{bmatrix} 2 & 1 & -3 \\ 5 & 4 & 1 \\ 2 & 6 & 7 \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} -\frac{22}{55} & \frac{25}{55} & -\frac{13}{55} \\ \frac{33}{55} & -\frac{20}{55} & \frac{17}{55} \\ -\frac{22}{55} & \frac{10}{55} & -\frac{3}{55} \end{bmatrix}$