Math 270 Linear Algebra

Chapter 3 Determinants

3.3 Cofactor Expansion

Definition Let $A = [a_{ij}]$ be an *n* x *n* matrix. Let M_{ij} be the $(n-1)\times(n-1)$ submatrix of *A* obtained by deleting the *i*th row and *j*th column of *A*. The determinant $\det(M_{ij})$ is called the **minor** of a_{ij}.

Definition The **cofactor** A_{ij} of a_{ij} is defined as $A_{ij} = (-1)^{i+j}\det(M_{ij})$.

Example $A = \begin{bmatrix} 2 & 1 & -3 \\ 5 & 4 & 1 \\ 2 & 6 & 7 \end{bmatrix}$

$$\det(M_{13}) = \begin{vmatrix} 5 & 4 \\ 2 & 6 \end{vmatrix} = 30 - 8 = 22$$

$$\det(M_{21}) = \begin{vmatrix} 1 & -3 \\ 6 & 7 \end{vmatrix} = 7 + 18 = 25$$

$$A_{13} = (-1)^{1+3}\det(M_{13}) = (1)(22) = 22$$

$$A_{21} = (-1)^{2+1}\det(M_{21}) = (-1)(25) = -25$$

Pattern of Signs

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} \qquad \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

Theorem Let $A = [a_{ij}]$ be *n* x *n*.

$\det(A) = a_{i1}A_{i1} + a_{i2}A_{i2} + \dots + a_{in}A_{in}$ expansion along the *i*th row

$\det(A) = a_{1j}A_{1j} + a_{2j}A_{2j} + \dots + a_{nj}A_{nj}$ expansion along the *j*th column

Example

$$\begin{vmatrix} 2 & 1 & -3 \\ 5 & 4 & 1 \\ 2 & 6 & 7 \end{vmatrix} = (-1)^{2+1}\cdot 5\begin{vmatrix} 1 & -3 \\ 6 & 7 \end{vmatrix} + (-1)^{2+2}\cdot 4\begin{vmatrix} 1 & -3 \\ 6 & 7 \end{vmatrix} + (-1)^{2+3}\begin{vmatrix} 2 & 1 \\ 2 & 6 \end{vmatrix}$$

$$= -5(7+18)+4(14+6)\text{-}1(12-2)\text{=-}55$$

Proof for *n* = 3:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

$$\begin{aligned} \det(A) &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12} \quad (*) \\ &= a_{11}(a_{22}a_{33} - a_{32}a_{23}) + a_{12}(a_{23}a_{31} - a_{33}a_{21}) + a_{13}(a_{21}a_{32} - a_{31}a_{22}) \end{aligned}$$

$$A_{11} = (-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{32}a_{23}$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = -(a_{21}a_{33} - a_{31}a_{23})$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}a_{32} - a_{31}a_{22}$$

Thus,

$\det(A) = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$ expansion of A along the first row

Writing (*) as

$$\begin{aligned} \det(A) &= a_{12}(a_{23}a_{31} - a_{33}a_{21}) + a_{22}(a_{11}a_{33} - a_{31}a_{13}) + a_{32}(a_{13}a_{21} - a_{23}a_{11}) \\ &= a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32} \end{aligned}$$

Exercise #4

$$\det(A) = \begin{vmatrix} 1 & 0 & 3 & 0 \\ 2 & 1 & -4 & -1 \\ 3 & 2 & 4 & 0 \\ 0 & 3 & -1 & 0 \end{vmatrix}$$

Expand about the first row:

$$\begin{aligned} \det(A) &= (-1)^{1+1}(1)\begin{vmatrix} 1 & -4 & -1 \\ 2 & 4 & 0 \\ 3 & -1 & 0 \end{vmatrix} + (-1)^{1+2}(0)\begin{vmatrix} 2 & -4 & -1 \\ 3 & 4 & 0 \\ 0 & -1 & 0 \end{vmatrix} \\ &\quad + (-1)^{1+3}(3)\begin{vmatrix} 2 & 1 & -1 \\ 3 & 2 & 0 \\ 0 & 3 & 0 \end{vmatrix} + (-1)^{1+4}(0)\begin{vmatrix} 2 & 1 & -4 \\ 3 & 2 & 4 \\ 0 & 3 & 1 \end{vmatrix} \\ &= \begin{vmatrix} 1 & -4 & -1 \\ 2 & 4 & 0 \\ 3 & -1 & 0 \end{vmatrix} + 3\begin{vmatrix} 2 & 1 & -1 \\ 3 & 2 & 0 \\ 0 & 3 & 0 \end{vmatrix} \\ &\overset{\text{3rd row}}{=} (-1)^{1+3}(-1)\begin{vmatrix} 2 & 4 \\ 3 & -1 \end{vmatrix} + 3(-1)^{1+3}\begin{vmatrix} 3 & 2 \\ 0 & 3 \end{vmatrix} \\ &= (-1)(-2-12) + 3(9) = 14 + 27 = 41 \end{aligned}$$

OR: Expand about the 4^{th} column from the beginning!

$$\det(A) = (-1)^{2+4}(-1)\begin{vmatrix} 1 & 0 & 3 \\ 3 & 2 & 4 \\ 0 & 3 & -1 \end{vmatrix}$$

$$\overset{\text{1st row}}{=} (-1)(-1)^{1+1}(1)\begin{vmatrix} 2 & 4 \\ 3 & -1 \end{vmatrix} + (-1)^{1+2}(0)\left|\ \right| + (-1)^{1+3}(3)\begin{vmatrix} 3 & 2 \\ 0 & 3 \end{vmatrix}$$

$$= (-1)(-2-12) + 3(9-0) = 41$$

Note: Expand about the row or column with the most number of zeros.

Exercise #12 Find all values of t for which

$$\begin{vmatrix} t-1 & 0 & 1 \\ -2 & t+2 & -1 \\ 0 & 0 & t+1 \end{vmatrix} = 0$$

Solution:

Expand about the 3rd row:

$$(-1)^{3+3}(t+1)\begin{vmatrix} t-1 & 0 \\ -2 & t+2 \end{vmatrix} = 0$$

$$(t+1)(t-1)(t+2) = 0$$

$$t = -1, 1, -2$$