Math 270 Linear Algebra

Chapter 3 Determinants

3.2 Properties of Determinants

Theorem $\det(A) = \det\left(A^T\right)$

Ex: $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2$

$\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = 4 - 6 = -2$

Theorem If B results from A by interchanging two rows (columns) of A**,** then $\det(B) = -\det(A)$

Ex: $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2$

$\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = 6 - 4 = 2$

Theorem If 2 rows (columns) of A are equal, then $\det(A) = 0$.

‘ Ex: $\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = 2 - 2 = o$

Theorem If a row (column) of A consists entirely of zeros, then $\det(A) = 0$.

Ex: $\left|\begin{matrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 4 & 5 & 6 \end{matrix}\right| \begin{matrix} 1\,2 \\ 0\,0 \\ 4\,5 \end{matrix} = 0$

Theorem If B is obtained from A by multiplying a row (column) of A by a real number k, then $\det(B) = k\det(A)$.

Ex: $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2$

$\xrightarrow{3R_1} \begin{vmatrix} 3 & 6 \\ 3 & 4 \end{vmatrix} = 12 - 18 = -6 = 3(-2)$

Ex: $\begin{vmatrix} 1 & 2 & 3 \\ -1 & 1 & 2 \\ 5 & 20 & 25 \end{vmatrix} \xrightarrow{1/5R_3} \begin{vmatrix} 1 & 2 & 3 \\ -1 & 1 & 2 \\ 1 & 4 & 5 \end{vmatrix}$

Big numbers! Easier

Theorem If $A \xrightarrow{\text{Type III}} B$, then $\det(B) = \det(A)$.

Ex: $\begin{vmatrix} 1 & 2 & 3 \\ -1 & 1 & 5 \\ 4 & -2 & 1 \end{vmatrix} \xrightarrow{2R_1+R_2} \begin{vmatrix} 1 & 2 & 3 \\ 1 & 5 & 11 \\ 4 & -2 & 1 \end{vmatrix} \Rightarrow \det = 47$

Theorem If $A = \left[a_{ij}\right]$ is upper (lower) triangular, then $\det(A) = a_{11}a_{22}...a_{nn}$.

Ex: $\begin{vmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{vmatrix} = (1)(2)(3) = 6$

So, another way to compute $\det(A)$ is via reduction to triangular form.

Exercise #2a

$$\begin{vmatrix} 2 & -2 \\ 3 & -1 \end{vmatrix} = 2\begin{vmatrix} 1 & -1 \\ 3 & -1 \end{vmatrix} = 2\begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix} = 2(1)(2) = 4$$

Exercise #2d

$$\begin{vmatrix} 4 & -3 & 5 \\ 5 & 2 & 0 \\ 2 & 0 & 4 \end{vmatrix} = 2\begin{vmatrix} 4 & -3 & 5 \\ 5 & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix} = -2\begin{vmatrix} 1 & 0 & 2 \\ 5 & 2 & 0 \\ 4 & -3 & 5 \end{vmatrix} = -2\begin{vmatrix} 1 & 0 & 2 \\ 0 & 2 & -10 \\ 0 & -3 & -3 \end{vmatrix} = 6\begin{vmatrix} 1 & 0 & 2 \\ 0 & 2 & -10 \\ 0 & 1 & 1 \end{vmatrix} = 6\begin{vmatrix} 1 & 0 & 2 \\ 0 & 2 & -10 \\ 0 & 0 & 6 \end{vmatrix} = 6(1)(2)(6) = 72$$

Lemma If E is an elementary matrix, then $\det(EA) = \det(E)\det(A)$ and
$\det(AE) = \det(A)\det(E)$.

E Type I: $\det(EA) = -\det(A) \Rightarrow \det(E) = -1$
E Type II: $\det(EA) = k\det(A) \Rightarrow \det(E) = k$
E Type III: $\det(EA) = \det(A) \Rightarrow \det(E) = 1$

Remark If $B = E_rE_{r-1}...E_2E_1A$, then $\det(B) = \det(E_r)\det(E_{r-1})...\det(E_2)\det(E_1)\det(A)$.

Theorem If A is n x n, then A is nonsingular if and only if $\det(A) \neq 0$.

Proof:

(" $\Rightarrow$ ") A nonsingular $\Rightarrow$ A is a product of elementary matrices $A = E_1E_2...E_k$
$\Rightarrow \det(A) = \det(E_1E_2...E_k) = \det(E_1)\det(E_2)...\det(E_k) \neq 0$

(" $\Leftarrow$ ") Let $\det(A) \neq 0$. Show A is nonsingular.
A is singular $\Rightarrow$ A is row equivalent to a matrix B that has a row of zeros
$\Rightarrow A = E_1E2...E_rB,\ E_i$ elementary matrices

$$\Rightarrow \det(A) = \det(E_1 \ldots E_r B) = \det(E_1)\det(E_2)\ldots\det(E_r)\underbrace{\det(B)}_{=0}$$

$$\Rightarrow \det(A) = 0, \text{ a contradiction}$$

Corollary If A is n x n then $A\vec{x} = \vec{0}$ has a nontrivial solution $\Leftrightarrow \det(A) = 0$.

Theorem If A and B are n x n matrices, then $\det(AB) = \det(A)\det(B)$.

Example $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 2 \\ -5 & -1 \end{bmatrix}$

$$|A| = 4 - 6 = -2; \qquad |B| = 1 + 10 = 11$$

$$AB = \begin{bmatrix} -16 & -1 \\ -22 & 0 \end{bmatrix} = -22 = |A||B|$$

Corollary If A is nonsingular, then $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

Example $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

$$[A \mid I] = \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 4 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & -2 & -2 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|cc} 1 & 0 & -2 & 3/2 \\ 0 & 1 & 1 & -1/2 \end{array}\right] \Rightarrow A^{-1} = \begin{bmatrix} -2 & 3/2 \\ 1 & -1/2 \end{bmatrix}$$

$$|A| = 4 - 6 = -2, \quad |A^{-1}| = (-2)\left(-\frac{1}{2}\right) - 1 \cdot \frac{3}{2} = -\frac{1}{2}$$

Corollary If A and B are similar matrices, then $\det(A) = \det(B)$.
SIMILAR MATRICES NOT DEFINED UNTIL CHAPTER 6 PAGE 410!!!

Remark In general, $\det(A + B) \neq \det(A) + \det(B)$
but if the kth row (column) of a matrix C = kth row (col) of A + kth row of B and all the other entries are the same, then $\det(C) = \det(A) + \det(B)$.

Example

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$$

$$\det(A) = -2, \quad \det(B) = -3, \quad \det(C) = -5 = \det(A) + \det(B)$$

Exercise #4 If $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -2$, find $\begin{vmatrix} a_1 - \frac{1}{2}a_3 & a_2 & a_3 \\ b_1 - \frac{1}{2}b_3 & b_2 & b_3 \\ c_1 - \frac{1}{2}c_3 & c_2 & c_3 \end{vmatrix}$.

Solution: -2