Math 270 Linear Algebra

Chapter 2 Solving Linear Systems

2.4 Equivalent Matrices

Definition If A and B are two m x n matrices, then A is **equivalent** to B if we obtain B from A by a finite sequence of elementary row or elementary column operations.

Remarks

1. A is equivalent to A.
A is equivalent to $B \Rightarrow B$ is equivalent to A
A is equivalent to B and B is equivalent to $C \Rightarrow A$ is equivalent to C

Proof: Exercise 1

2. A and B are row equivalent $\Rightarrow A$ and B are equivalent

Proof: Exercise 4

Theorem If A is any nonzero m x n matrix, then A is equivalent to a partitioned matrix of the form

$$\begin{bmatrix} I_r & {}_rO_{n-r} \\ {}_{m-r}O_n & {}_{m-r}O_{n-r} \end{bmatrix}$$

Proof: Read.

Example 1 #6 p.129

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{\text{pivot on } a_{11}} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{\text{pivot on } a_{22}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{-C_2+C_3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{-C_1+C_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Theorem Two m x n matrices A and B are equivalent if and only if $B = PAQ$ for some nonsingular matrices P and Q.

Proof:

$("\Rightarrow")$ A and B are equivalent

$\Rightarrow B = E_s \ldots E_2 E_1 A F_1 F_2 \ldots F_t$

Let $P = E_s \ldots E_2 E_1$ and $Q = F_1 F_2 \ldots F_t$

$("\Leftarrow")$ Exercise

Theorem An n x n matrix is nonsingular if and only if A is equivalent to I_n.

Proof:

$("\Rightarrow")$ A nonsingular $\Rightarrow A$ row equivalent to I_n

$\Rightarrow A$ equivalent to I_n

$("\Leftarrow")$ A equivalent to $I_n \Rightarrow I_n = E_r \ldots E_2 E_1 A F_1 F_2 \ldots F_s$

Let $E_r \ldots E_2 E_1 = P$ and $F_1 F_2 \ldots F_s = Q$. Then $I_n = PAQ$, where P and Q are nonsingular. Then $A = P^{-1} Q^{-1}$ which implies that A is nonsingular.