Math 270 Linear Algebra

Chapter 2 Solving Linear Systems

2.3 Elementary Matrices; Finding A^{-1}

Definition An *n* x *n* elementary matrix of type I, type II, or type III is a matrix obtained from the identity matrix I_n by performing a single elementary row or elementary column operation of type I, type II, or type III, respectively.

Example 1

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{matrix} R_1 \leftrightarrow R_2 \\ \text{type I} \end{matrix}$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad \begin{matrix} 3R_3 \\ \text{type II} \end{matrix}$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \quad \begin{matrix} -2R_2 + R_3 \\ \text{type III} \end{matrix}$$

Theorem Let *A* be an *m* x *n* matrix, and let an elementary row (column) operation of type **I**, **II**, or **III** be performed on *A* to yield a matrix *B.* Let *E* be the elementary matrix obtained from $I_m\left(I_n\right)$ by performing the same elementary row (column) operation as was performed on *A*. Then

$$B = EA \quad \left(B = AE\right)$$

Proof: Exercise 1

Example 2

$$A = \begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & -1 \\ 2 & 5 & 1 \end{bmatrix} \xrightarrow{-2R_3 + R_1} \begin{bmatrix} -3 & -13 & 0 \\ 0 & 1 & -1 \\ 2 & 5 & 1 \end{bmatrix} = B$$

$$E = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Pre-multiply *A* by *E*:

$$EA = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & -1 \\ 2 & 5 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -13 & 0 \\ 0 & 1 & -1 \\ 2 & 5 & 1 \end{bmatrix} = B$$

Theorem Let A and B be m x n matrices. Then A is row (column) equivalent to B if and only if there exist elementary matrices $E_1, E_2, ..., E_k$ such that

$$B = E_k E_{k-1} ... E_2 E_1 A$$
$$(B = A E_1 E_2 ... E_{k-1} E_k)$$

Proof: (for row equivalence)

("$\Rightarrow$") A row equivalent to B

$\Rightarrow$ B results from A by a sequence of elementary row operations

$\Rightarrow$ there exist elementary matrices $E_1, E_2, ..., E_k$ s.t. $B = E_k E_{k-1} ... E_2 E_1 A$.

("$\Leftarrow$") $B = E_k E_{k-1} ... E_2 E_1 A$

$\Rightarrow$ B results from A by a sequence of elementary row operations

$\Rightarrow$ A is row equivalent to B

Theorem An elementary matrix E is nonsingular and its inverse is an elementary matrix of the same type.

To illustrate for n=3:

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \text{ Then } E_1^{-1} = E_1.$$

Check:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad c \text{ real}, \quad c \neq 0$$

$$\text{Then } E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/c & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{bmatrix}$$

$$\text{Then } E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -c & 1 \end{bmatrix}$$

Let E_1 be type **I**. Then $E_1^{-1} = E_1$.

Let E_2 be type **II**, i.e. multiply the ith row of *I* by *c*, $c \neq 0$. Let E_2^* be obtained by multiplying the ith row of *I* by $1/c$, $c \neq 0$. Then $E_2 E_2^* = I$.

Let E_3 be type III, i.e. add to row *i* *c* times row *j*, then E_3^{-1} is obtained by adding to row *i* -*c* times row *j*.

Note: The theorem says an elementary row operation can be "undone" by another elementary row operation of the same type.

To find A^{-1}:

Lemma Let *A* be an *n* x *n* matrix and let the homogeneous system $A\vec{x} = \vec{0}$ have only the trivial solution $\vec{x} = \vec{0}$. Then *A* is row equivalent to I_n.

Proof: read

Theorem *A* is nonsingular if and only if *A* is a product of elementary matrices.

Proof:

$("\Leftarrow")$ *A* is a product of elementary matrices $E_1, E_2 ..., E_k$

$\Rightarrow A = E_1 E_2 ... E_k$

Each E_i is nonsingular and the product of nonsingular matrices is nonsingular. Thus, *A* is nonsingular.

$("\Rightarrow")$ Let *A* be nonsingular. Then

$A\vec{x} = \vec{0} \Rightarrow A^{-1}\left(A\vec{x}\right) = A^{-1}\vec{0} = \vec{0}$

$\Rightarrow I_n \vec{x} = \vec{0}$

$\Rightarrow \vec{x} = \vec{0}$

$\Rightarrow A\vec{x} = \vec{0}$ has only the trivial solution

$\Rightarrow A$ is row equivalent to I_n

$\Rightarrow I_n = E_k E_{k-1} ... E_2 E_1 A$ for some elementary matrices $E_1, E_2, ..., E_k$

$\Rightarrow A = \left(E_k E_{k-1} ... E_2 E_1\right)^{-1} = E_1^{-1} E_2^{-1} ... E_k^{-1}$

$\Rightarrow A$ is a product of elementary matrices

Corollary *A* is nonsingular if and only if *A* is row equivalent to I_n.

Proof:

$("\Leftarrow")$ *A* row equivalent to I_n

$\Rightarrow I_n = E_k E_{k-1} ... E_2 E_1 A$ where $E_1, E_2, ..., E_k$ are elementary matrices

$\Rightarrow A = \left(E_k E_{k-1} ... E_2 E_1\right)^{-1} = E_1^{-1} E_2^{-1} ... E_k^{-1}$, i.e *A* is a product of elementary matrices

$\Rightarrow A$ is nonsingular

$("\Rightarrow")$ A is nonsingular

$\Rightarrow A = E_k E_{k-1} ... E_2 E_1$

$\Rightarrow A = AI_n = E_k E_{k-1} ... E_2 E_1 I_n$

$\Rightarrow A$ is row equivalent to I_n

Remarks

1) The lemma and corollary imply the following:

$A\vec{x} = \vec{0}$ has only the trivial solution $\vec{x} = \vec{0} \Rightarrow A$ is nonsingular .

2) Conversely, consider $A\vec{x} = \vec{0}$, A $n \times n$, and let A be nonsingular. Then A^{-1} exists and

$$A^{-1}\left(A\vec{x}\right) = A^{-1}\vec{0} = \vec{0}$$
$$\left(A^{-1}A\right)\vec{x} = \vec{0}$$
$$I_n\vec{x} = \vec{0}$$
$$\vec{x} = \vec{0}$$

Thus, $A\vec{x} = \vec{0}$ has only the trivial solution and we proved:

Theorem Let A be n x n. $A\vec{x} = \vec{0}$ has a nontrivial solution if and only if A is singular.

Example 3

$$A = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$$

$$Ax = 0: \left[\begin{array}{cc|c} 1 & -2 & 0 \\ -2 & 4 & 0 \end{array}\right] \xrightarrow{\text{pivot on } a_{11}} \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{c} x_1 = 2r \\ x_2 = r,\ r \text{ real} \end{array}$$

$\Rightarrow$ system has a nontrivial solution

Remark In Section 1.5, we proved that:

A is nonsingular $\Rightarrow A\vec{x} = \vec{b}$ has a unique solution for every n-vector b.

The converse is also true, i.e.

$A\vec{x} = \vec{b}$ has a unique solution $\Rightarrow A$ is nonsingular.

Note For an n x n matrix A, the following are equivalent (tfae):

1. A is nonsingular.
2. $Ax = 0$ has only the trivial solution
3. A is row (column) equivalent to I_n
4. $Ax = b$ has a unique solution for every n-vector b.
5. A is a product of elementary matrices.

Remark In one of the proofs we saw that if A is nonsingular, then

$$A = E_1^{-1}E_2^{-1}...E_{k-1}^{-1}E_k^{-1}.$$

This implies that

$$A^{-1} = \left(E_1^{-1}E_2^{-1}...E_{k-1}^{-1}E_k^{-1}\right)^{-1} = E_kE_{k-1}...E_2E_1.$$

This provides an algorithm for finding A^{-1}:

$$\left[A \mid I_n\right] \xrightarrow{\text{row operations}} \left[I_n \mid A^{-1}\right]$$

Exercise 8 p. 124

Given $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix}$. Find A^{-1}.

Solution:

$$\left[A \mid I_3\right] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\text{pivot on } a_{11}} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\text{pivot on } a_{22}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 3/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\text{pivot on } a_{33}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 3/2 & 1/2 & -3/2 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \Rightarrow A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 3/2 & 1/2 & -3/2 \\ -1 & 0 & 1 \end{bmatrix}$$

Verify: $AA^{-1} = I_3$

How can we tell if A is singular?

Theorem A is singular if and only if A is row equivalent to a matrix B that has a row of zeros.

Proof:

("$\Leftarrow$") Let A be row equivalent to a matrix B that has a row of zeros

$\Rightarrow$ B is singular (by Exercise 46 Section 1.5)

$B = E_kE_{k-1}...E_2E_1A$, where the E_i are elementary matrices

If A is nonsingular then B is nonsingular, a contradiction!

Thus, A is singular.

$(" \Rightarrow ")$ A is singular

$\Rightarrow A$ is <u>not</u> row equivalent to I_n

$\Rightarrow A$ is row equivalent to a matrix $B \neq I_n$, which is in r.r.e.f.

$\Rightarrow B$ must have a row of zeros (Exercise 9 Section 2.1)

Exercise 9d p. 125

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

Solution:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{pivot on } a_{11}} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\text{pivot on } a_{22}} \left[\begin{array}{ccc|ccc} 1 & 0 & & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & & & 1 \end{array}\right]$$

row of zeros! $\Rightarrow A$ singular

Theorem If A and B are n x n matrices such that $AB = I_n$, then $BA = I_n$. Thus, $B = A^{-1}$.

Proof: Read.

Remark The theorem says it suffices to check whether $AB = I_n$ <u>or</u> $BA = I_n$ (not both) to establish that $B = A^{-1}$.