Math 270 Linear Algebra

Chapter 2 Solving Linear Systems

2.2 Solving Linear Systems

Solving Linear Systems $A\vec{x}=\vec{b}$

Step 1 $\left[A \mid \vec{b}\right]$

$\downarrow$

$\left[C \mid \vec{d}\right]$ r.e.f. or r.r.e.f

$\downarrow$ Gaussian elimination $\qquad$ $\downarrow$ Gauss-Jordan reduction

Step 2 Do back substitution.

Example 3 Solve:

$$\begin{aligned} x_1+x_2+x_3&=1\\ x_1+x_2-2x_3&=3\\ 2x_1+x_2+x_3&=2 \end{aligned}$$

Solution:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1\\ 1 & 1 & -2 & 3\\ 2 & 1 & 1 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1\\ 0 & 0 & -3 & 2\\ 0 & -1 & -1 & 0 \end{array}\right]$$

$$\xrightarrow{R_2\leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1\\ 0 & -1 & -1 & 0\\ 0 & 0 & -3 & 2 \end{array}\right] \xrightarrow{(-1)R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1\\ 0 & 1 & 1 & 0\\ 0 & 0 & -3 & 2 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1\\ 0 & 1 & 1 & 0\\ 0 & 0 & -3 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1\\ 0 & 1 & 0 & 2/3\\ 0 & 0 & 1 & -2/3 \end{array}\right] \Rightarrow \begin{aligned} x_1&=1\\ x_2&=2/3\\ x_3&=-2/3 \end{aligned}$$

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$$\begin{aligned} x+y+2z+3w&=13\\ x-2y+z+w&=8\\ 3x+y+z-w&=1 \end{aligned}$$

Solution:

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 3 & 13\\ 1 & -2 & 1 & 1 & 8\\ 3 & 1 & 1 & -1 & 1 \end{array}\right] \xrightarrow{\text{pivot on } a_{11}} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 3 & 13\\ 0 & -3 & -1 & -2 & -5\\ 0 & -2 & -5 & -10 & -38 \end{array}\right]$$

$$\xrightarrow{(-2)R_3+R_2} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 3 & 13\\ 0 & 1 & 9 & 18 & 71\\ 0 & -2 & -5 & -10 & -38 \end{array}\right] \xrightarrow{\text{pivot on } a_{22}} \left[\begin{array}{cccc|c} 1 & 0 & -7 & -15 & -58\\ 0 & 1 & 9 & 18 & 71\\ 0 & 0 & 13 & 26 & 104 \end{array}\right]$$

$$\xrightarrow{\text{pivot on } a_{33}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -2\\ 0 & 1 & 0 & 0 & -1\\ 0 & 0 & 1 & 2 & 8 \end{array}\right] \Rightarrow \begin{aligned} x \qquad -w&=-2\\ y \qquad &=-1\\ z+2w&=8 \end{aligned} \Rightarrow \begin{aligned} x&=-2+r\\ y&=-1\\ z&=8-2r\\ w&=r \end{aligned}$$

Homogeneous Systems $A\vec{x} = \vec{0}$

Example 5

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{array}\right] \Rightarrow \begin{array}{c} x_1 = 0 \\ x_2 + x_4 = 0 \\ x_3 + 2x_4 = 0 \\ x_4 = r, \ \ r \text{ real} \end{array} \Rightarrow \begin{array}{c} x_1 = 0 \\ x_2 = -r \\ x_3 = -2r \\ x_4 = r, \ \ r \text{ real} \end{array}$$

Theorem A homogeneous system of *m* linear equations in *n* unknowns always has a nontrivial solution if $m < n$, i.e. if the number of unknowns exceeds the number of equations.

Example 6

$$\begin{array}{l} x_1 + 2x_2 + x_3 - x_4 = 0 \\ x_1 + x_2 \qquad\quad + x_4 = 0 \\ x_1 \qquad + 2x_3 \qquad = 0 \end{array}$$

Solution:

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 \end{array}\right] \xrightarrow{\text{pivot on } a_{11}} \left[\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 0 \\ 0 & -1 & -1 & 2 & 0 \\ 0 & -2 & 1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{\text{pivot on } a_{22}} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 3 & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 3 & -3 & 0 \end{array}\right]$$

$$\xrightarrow{\text{pivot on } a_{33}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right] \Rightarrow \begin{array}{c} x_1 = -2r \\ x_2 = r \\ x_3 = r \\ x_4 = r, \ r \text{ real} \end{array}$$

Relationship Between $A\vec{x} = \vec{b}$ **and** $A\vec{x} = \vec{0}$

Let $A\vec{x} = \vec{b}, \ \vec{b} \neq \vec{0}$, be a consistent linear system.

$\vec{x}_p$ a particular solution to $A\vec{x} = \vec{b}$

x_h solution to $A\vec{x} = \vec{0}$

Then $\vec{x}_p + \vec{x}_h$ is a solution to $A\vec{x} = \vec{b}$. Moreover, every solution to $A\vec{x} = \vec{b}$ $\left(\vec{b} \neq \vec{0}\right)$, can be written as $\vec{x}_p + \vec{x}_h$, where $\vec{x}_p$ is a particular solution to the given nonhomogeneous system and $\vec{x}_h$ is a solution to $A\vec{x} = \vec{0}$.

Proof: HW (Exercise 29)