Math 270 Linear Algebra

Chapter 2 Solving Linear Systems

2.1 Echelon Form of a Matrix

Suppose we have

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & 3 \end{array}\right] \qquad (*)$$

$$\Rightarrow \begin{cases} x_1 - x_2 + x_3 = 2 \\ \quad\; x_2 + x_3 = 5 \\ \qquad\quad x_3 = 3 \end{cases} \Rightarrow \begin{cases} x_3 = 3 \\ x_2 = 5 - x_3 = 5 - 3 = 2 \\ x_1 = 2 + x_2 - x_3 = 2 + 2 - 3 = 1 \end{cases}$$

Thus, the system is easy to solve if it looks like (*).

Definitions An m x n matrix A is said to be in **reduced row echelon form** if it satisfies the following properties:

- **(a)** All zero rows, if any, appear at the bottom of the matrix.
- **(b)** The first nonzero entry of a nonzero row is 1, called the **leading one** of its row.
- **(c)** For each nonzero row, the leading one appears to the right and below any leading ones in the preceding rows.
- **(d)** If a columns contains a leading one, then all the other entries in that column are zero.

If A satisfies only (a), (b), and (c), then A is said to be in **row echelon form**.
Similar definitions can be formulated **reduced column echelon form** and **column echelon form**.

Example 1

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix} \text{ are in rref}$$

$$C = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},\ D = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ are in ref}$$

Remark If A is an n x n matrix in rref and $A \neq I_n$, then A has a row consisting entirely of zeros.

Proof: (Exercise 3)

Definition An **elementary row (column) operation** in a matrix A is any one of the following operations:

- (a) **Type I**: Interchange any two rows (columns).
- (b) **Type II**: Multiply a row (column) by a nonzero number.
- (c) **Type III**: Add a multiple of one row (column) to another.

Example 2

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ -2 & 0 & 0 & 8 \\ -1 & 1 & 3 & -2 \end{bmatrix}$$

(a) $R_1 \leftrightarrow R_3$

$$\begin{bmatrix} -1 & 1 & 3 & -2 \\ -2 & 0 & 0 & 8 \\ 1 & 0 & 2 & 3 \end{bmatrix}$$

(b) $\frac{1}{2} R_2$

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ -1 & 0 & 0 & 4 \\ -1 & 1 & 3 & -2 \end{bmatrix}$$

(c) $-2R_2 + R_3$

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ -2 & 0 & 0 & 8 \\ 3 & 1 & 3 & -18 \end{bmatrix}$$

Note: R_2 did <u>not</u> change.

Definition An m x n matrix B is said to be **<u>row (column) equivalent</u>** to an m x n matrix A if B can be obtained by applying a finite sequence of row (column) operations to A.

Example 3

$$A = \begin{bmatrix} 2 & 1 & 0 & 3 \\ 1 & -1 & 1 & 5 \\ 2 & 4 & 3 & 1 \end{bmatrix}$$

$$\xrightarrow{2R_2} \begin{bmatrix} 2 & 1 & 0 & 3 \\ 2 & -2 & 2 & 10 \\ 2 & 4 & 3 & 1 \end{bmatrix} = B$$

$$\xrightarrow{-R_3 + R_1} \begin{bmatrix} 0 & -3 & -3 & 2 \\ 2 & -2 & 2 & 10 \\ 2 & 4 & 3 & 1 \end{bmatrix} = C$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 2 & -2 & 2 & 10 \\ 0 & -3 & -3 & 2 \\ 2 & 4 & 3 & 1 \end{bmatrix} = D$$

D is row equivalent to A.

Remarks:

1) Every matrix is row equivalent to itself.
2) If B is row equivalent to A, then A is row equivalent to B,
3) If C is row equivalent to B and
B is row equivalent to A
then C is row equivalent to A.

Theorem Every nonzero m x n matrix $A = \left[a_{ij} \right]$ is row (column) equivalent to a matrix in row (column) echelon form or reduced row (column) echelon form.

Example

$$A = \begin{bmatrix} 1 & -2 & 0 & 2 \\ 2 & -3 & -1 & 5 \\ 1 & 3 & 2 & 5 \\ 1 & 1 & 0 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 0 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 5 & 2 & 3 \\ 0 & 3 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 0 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 7 & -2 \\ 0 & 0 & 3 & -3 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{3}R_4} \begin{bmatrix} 1 & -2 & 0 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 7 & -2 \\ 0 & 0 & 1 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 0 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 7 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 0 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{5}R_4} \begin{bmatrix} 1 & -2 & 0 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

`

Note: I did partial pivoting to get ref, i.e., I did not change the numbers above the 1 to 0's.

If we want rref

$$\begin{bmatrix} 1 & -2 & 0 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Note: Full pivoting gives rref right away.

Theorem Let $A\vec{x} = \vec{b}$ and $C\vec{x} = \vec{d}$ be two linear systems of m equations in n unknowns. If $\left[A \,\vert\, \vec{b} \right]$ and $\left[C \,\vert\, \vec{d} \right]$ are row equivalent, then the linear systems are equivalent, i.e. they have exactly the same solutions.

Corollary If A and C are row equivalent m x n matrices, then the homogeneous systems $A\vec{x} = \vec{0}$ and $C\vec{x} = \vec{0}$ are equivalent.