Math 270 Linear Algebra

Chapter 1 Linear Equations and Matrices

1.6 Matrix Transformations

$$\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \quad \text{2-vector}$$

$$\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \quad \text{3-vector}$$

A is m x n matrix; $\vec{u}$ n-vector $\Rightarrow A\vec{u}$ is an m-vector

Define $f : \mathbb{R}^n \to \mathbb{R}^m$ by $f(\vec{u}) = A\vec{u}$. f is called a **matrix transformation**. $f(\vec{u})$ is called the **image** of $\vec{u}$ and the set of all images is called the **range** of f.

Example 1 $f : \mathbb{R}^2 \to \mathbb{R}^2$

$$f(\vec{u}) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \vec{u}$$

Then, $f\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$.

Example 2 $f : \mathbb{R}^3 \to \mathbb{R}^2$

$$f(\vec{u}) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix} \vec{u}$$

Then $f\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}$

Example 3 Exercise 10

$f : \mathbb{R}^2 \to \mathbb{R}^3$

$$f(\vec{x}) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \vec{x}$$

Determine whether $\vec{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is in the range of f.

Solution:

To find a vector $\vec{x}$, if possible, s.t.

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \begin{array}{rcl} x_1 + 2x_2 & = & 1 \\ x_2 & = & 1 \Rightarrow \varnothing \Rightarrow \text{NO} \\ x_1 + x_2 & = & 1 \end{array}$$

Example 4 p. 58 $f:\mathbb{R}^2 \to \mathbb{R}^2$

$$f(\vec{u}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \vec{u}$$

Then $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$

f is a **reflection** wrt the *x*-axis in $\mathbb{R}^2$.

Example 5 p. 58 $f:\mathbb{R}^3 \to \mathbb{R}^2$

$$f(\vec{u}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \vec{u}$$

$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

f is a **projection** into the *xy*-plane

Note that since $f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}$, then infinitely many 3-vectors have the same image.

Example 5b $f:\mathbb{R}^3 \to \mathbb{R}^3$

$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

Example 6 p. 59 $f:\mathbb{R}^3 \to \mathbb{R}^3$

$$f(\vec{u}) = \begin{bmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r \end{bmatrix} \vec{u}$$

$$f(\vec{u}) = r\vec{u}$$

$r > 1$, *f* is called a **dilation**

$0 < r < 1$, *f* is called a **contraction**

Similarly, if $g:\mathbb{R}^2 \to \mathbb{R}^2$

$$g(\vec{u}) = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \vec{u}$$

is a dilation if $r > 1$

contraction if 0<r<1

Read Examples 7 and 8