Math 270 Linear Algebra

Chapter 1 Linear Equations and Matrices

1.5 Special Types of Matrices and Partitioned Matrices

We have the following special n x n matrices:

diagonal matrix $a_{ij} = 0$ for $i \neq j$, i.e. the terms off the main diagonal are all zero

scalar matrix diagonal matrix whose diagonal elements are all equal

identity matrix diagonal matrix whose diagonal elements are all 1

Example 1

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A, B, I_3 diagonal
B, I_3 scalar
I_3 identity

Note:

1. If A is m x n, then $AI_n = A$ and $I_m A = A$.
2. If A is a scalar matrix, then $A = rI_n$ for some scalar r.

Proof: HW (Exercise 1)

To define the powers of a matrix:

Let p be a positive integer. Define

$$A^p = \underbrace{A \cdot A \cdot \ldots \cdot A}_{p \text{ factors}}.$$

If A is n x n, define

$$A^0 = I_n .$$

For nonnegative integers p and q:

$$A^p A^q = A^{p+q}$$

$$\left(A^p\right)^q = A^{pq}$$

Proofs: HW (Exercise 8)

Note that $(AB)^p = A^p B^p$ does <u>not</u> hold for square matrices unless $AB = BA$. (HW: Exercise 9)

Still more on special *n* x *n* matrices:

<u>upper triangular</u> $a_{ij} = 0$ for $i > j$
<u>lower triangular</u> $a_{ij} = 0$ for $i < j$

Note A diagonal matrix is both upper and lower triangular.

<u>symmetric matrix</u> $A^T = A$
<u>skew-symmetric matrix</u> $A^T = -A$

Example 3

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 5 & 2 \\ 0 & 2 & 4 \end{bmatrix} \text{ symmetric}$$

$$B = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 2 \\ -1 & -2 & 0 \end{bmatrix} \text{ skew-symmetric}$$

Note:

1. A symmetric or skew-symmetric $\Rightarrow$ A square.
2. A symmetric $\Rightarrow$ the entries of A are symmetric wrt the main diagonal of A
3. A symmetric $\Leftrightarrow$ $a_{ij} = a_{ji}$; A skew-symmetric $a_{ij} = -a_{ji}$.
4. A skew-symmetric $\Rightarrow$ the entries on the main diagonal of A are all zero.
5. A is n x n $\Rightarrow$ $A = S + K$, where S is a symmetric matrix and K is a skew-symmetric matrix. Moreover, this decomposition is unique. (HW: Exercise 29)

A **<u>submatrix</u>** of a matrix A is obtained by crossing out some, but not all, of the rows or columns of A.

Example 4

$$A = \begin{bmatrix} 1 & 4 & 3 \\ -1 & 2 & 5 \\ 2 & 0 & 1 \\ 3 & -2 & 0 \end{bmatrix} \quad \text{submatrix} \begin{bmatrix} 4 & 3 \\ 2 & 5 \\ -2 & 0 \end{bmatrix}$$

<u>partitioned matrix</u> draw horizontal lines between rows and vertical lines between columns

Example 5

$$A = \left[\begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

or

$$A = \left[\begin{array}{cc|c|cc} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}$$

Example 6 The augmented matrix of the system $A\vec{x}=\vec{b}$, $\left[A \mid \vec{b}\right]$, is a partitioned matrix.

Remarks

1. If A and B are both m x n matrices that are partitioned in the same way, then $A + B$ is obtained by simply adding the corresponding submatrices of A and B.
2. Similarly, if A is a partitioned matrix, then the scalar multiple cA is obtained by forming the scalar multiple of each submatrix.

Example 7 **block multiplication**

$$A=\left[\begin{array}{cc|cc} 0 & 1 & 2 & 0 \\ 3 & 2 & -1 & 2 \\ \hline 1 & 0 & 1 & 0 \\ 0 & 2 & -3 & 1 \end{array}\right]=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

and

$$\text{Let } B=\left[\begin{array}{ccc|cc} 1 & 2 & -1 & 1 & 0 \\ 2 & 0 & 3 & 0 & -1 \\ \hline -1 & 2 & 0 & 1 & 3 \\ 1 & 1 & 2 & 0 & -1 \end{array}\right]=\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

Then

$$AB=\begin{bmatrix} A_{11}B_{11}+A_{12}B_{21} & A_{11}B_{12}+A_{12}B_{22} \\ A_{21}B_{11}+A_{22}B_{21} & A_{21}B_{12}+A_{22}B_{22} \end{bmatrix}=\ldots$$

Example 8 **adjoining matrices** – reverse of partitioning

$$\text{Let } A=\begin{bmatrix} 1 \\ 2 \end{bmatrix},\ B=\begin{bmatrix} 0 & -1 & 5 \end{bmatrix},\ C=\begin{bmatrix} 0 & 2 & 1 \\ 1 & -1 & 3 \end{bmatrix}$$

Then

$$\begin{bmatrix} A & C \end{bmatrix}=\left[\begin{array}{c|ccc} 1 & 0 & 2 & 1 \\ 2 & 1 & -1 & 3 \end{array}\right]$$

$$\begin{bmatrix} C \\ B \end{bmatrix}=\left[\begin{array}{ccc} 0 & 2 & 1 \\ 1 & -1 & 3 \\ \hline 0 & -1 & 5 \end{array}\right]$$

$$\begin{bmatrix} C^T & B^T \end{bmatrix}=\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 2 & -1 & -1 \\ 1 & 3 & 5 \end{array}\right]$$

Definition An n x n matrix A is called **nonsingular**, or **invertible**, if there exists an n x n matrix B s.t. $AB = BA = I_n$; B is called an **inverse** of A. Otherwise, A is called **singular** or **noninvertible**.

Remark It will be shown later in 2.2 that if $AB = I_n$ then $BA = I_n$. Thus, we need only verify that $AB = I_n$ to show that B is an inverse of A.

Example 9

$$A=\begin{bmatrix}1 & 2\\ 3 & -1\end{bmatrix},\quad B=\begin{bmatrix}\frac{1}{7} & \frac{2}{7}\\ \frac{3}{7} & -\frac{1}{7}\end{bmatrix}$$

$$AB=BA=I_2$$

Theorem The inverse of a matrix, if it exists, is unique.

Proof:

Let B and C be inverses of A. Then

$$AB=BA=I_n \text{ and}$$
$$AC=CA=I_n$$
$$B=BI_n=B(AC)=(BA)C=I_nC=C$$

Thus, the inverse of A, if it exists, is unique.

Notation A^{-1} inverse of A; $AA^{-1}=A^{-1}A=I$

Finding A^{-1}

Example 10

Let $A=\begin{bmatrix}1 & 3\\ 2 & 4\end{bmatrix},\ A^{-1}=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$

Then $AA^{-1}=\begin{bmatrix}a+3c & b+3d\\ 2a+4c & 2b+4d\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

$$\Rightarrow \begin{array}{ll} a+3c=1 & (1)\\ b+3d=0 & (2)\\ 2a+4c=0 & (3)\\ 2b+4d=1 & (4)\end{array}$$

$\Rightarrow a=-2, b=3/2, c=1, d=-1/2$

Thus, $A^{-1}=\begin{bmatrix}-2 & 3/2\\ 1 & -1/2\end{bmatrix}$

Example 11

$$A=\begin{bmatrix}1 & -2\\ -2 & 4\end{bmatrix},\quad A^{-1}=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$$

$$AA^{-1}=\begin{bmatrix}a-2c & b-2d\\ -2a+4c & -2b+4d\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$\Rightarrow \varnothing \Rightarrow A$ is singular

<u>Properties of</u> A^{-1}

Theorem If A and B are both nonsingular n x n matrices, then AB is nonsingular and

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Proof:

$$\begin{aligned}(AB)(B^{-1}A^{-1}) &= A(BB^{-1})A^{-1} \\ &= (AI_n)A^{-1} \\ &= AA^{-1} \\ &= I_n\end{aligned}$$

Similarly, $(B^{-1}A^{-1})(AB) = I_n$.
Thus, AB is nonsingular and by the uniqueness of the inverse of a matrix, $(AB)^{-1} = B^{-1}A^{-1}$.

Corollary If $A_1, A_2, ..., A_r$ are n x n nonsingular matrices, then $A_1 \cdot A_2 \cdot ... \cdot A_r$ is nonsingular and $(A_1 \cdot A_2 \cdot ... \cdot A_r)^{-1} = A_r^{-1}A_{r-1}^{-1}...A_1^{-1}$.

Proof: HW (Exercise 44)

Theorem If A is nonsingular, then A^{-1} is nonsingular and $(A^{-1})^{-1} = A$.

Proof: HW (Exercise 45)

Theorem If A is nonsingular, then A^T is nonsingular and $(A^T)^{-1} = (A^{-1})^T$.

Proof: $AA^{-1} = I_n$

$$(AA^{-1})^T = I_n^T$$

$$(A^{-1})^T A^T = I_n$$

Similarly, $A^T(A^{-1})^T = I_n$.

Thus, $(A^T)^{-1} = (A^{-1})^T$.

Remarks

1. If A is nonsingular, then
   $AB = AC \Rightarrow B = C$ (HW: Exercise 50)
   $AB = O_n \Rightarrow B = O_n$ (HW: Exercise 51)

2. If A is a symmetric nonsingular matrix then A^{-1} is symmetric. (HW: Exercise 54)

Linear Systems and Inverses

A is n x n
$Ax = b$ (*) system of n linear equations in n unknowns

Suppose A is nonsingular. Then A^{-1} exists and

$$
\begin{aligned}
A^{-1}(Ax) &= A^{-1}b \\
(A^{-1}A)x &= A^{-1}b \\
I_n x &= A^{-1}b \\
x &= A^{-1}b \text{ is a solution to } (*) \text{ and the solution is unique}
\end{aligned}
$$

Thus, A is nonsingular $\Rightarrow x = A^{-1}b$ is the unique solution to (*).

Example 12 Use matrix A from Example 10

$$
\begin{aligned}
&Ax = b \\
&\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} x = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \\
&\text{Then } x = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 3 \end{bmatrix} \\
&\quad = \begin{bmatrix} -2 & 3/2 \\ 1 & -1/2 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix} \\
&\quad = \begin{bmatrix} -7/2 \\ 5/2 \end{bmatrix}
\end{aligned}
$$

Example 13 If we change the RHS b to $b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, then $x = \begin{bmatrix} -2 & 3/2 \\ 1 & -1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.