Math 270 Linear Algebra

Chapter 1 Linear Equations and Matrices

1.4 Algebraic Properties of Matrix Operations

Theorem <u>Properties of Matrix Addition</u>

Let A, B, and C be m x n matrices.

(a) $A+B=B+A$

(b) $A+(B+C)=(A+B)+C$

(c) There is a unique matrix O such that

$$A+O=A \qquad (1)$$

for any m x n matrix A. The matrix O is called the m x n **<u>zero matrix</u>**.

(d) For each m x n matrix A, there is a unique m x n matrix D such that

$$A+D=O \qquad (2)$$

We write $D=-A$, so (2) can be written as

$$A+(-A)=O.$$

$-A$ is called the **<u>negative</u>** of A. $-A=(-1)A$.

Proof:

(a) Let $A=\left[a_{ij}\right]$, $B=\left[b_{ij}\right]$, $A+B=C=\left[c_{ij}\right]$, $B+A=D=\left[d_{ij}\right]$

To show $c_{ij}=d_{ij}$ for all i, j.

Now, $c_{ij}=a_{ij}+b_{ij}$

$=b_{ij}+a_{ij}$ since a_{ij} and b_{ij} are real

$=d_{ij}$

(b) HW (Exercise 1)

(c) Let $U=\left[u_{ij}\right]$. Then

$$\begin{aligned} A+U=A &\Leftrightarrow a_{ij}+u_{ij}=a_{ij} \\ &\Leftrightarrow u_{ij}=0 \\ &\Leftrightarrow U=O \end{aligned}$$

(d) HW (Exercise 2)

Example 1

$$A=\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad O=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A+O=\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}+\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}=A$$

Example 2

$$A=\begin{bmatrix}1 & 2\\ 3 & -4\\ -5 & 6\end{bmatrix},\quad -A=\begin{bmatrix}-1 & -2\\ -3 & 4\\ 5 & -6\end{bmatrix}$$

$$A+(-A)=O$$

Theorem <u>Properties of Matrix Multiplication</u>

If A, B, and C are matrices of appropriate sizes, then

(a) $A(BC)=(AB)C$

(b) $(A+B)C=AC+BC$

(c) $C(A+B)=CA+CB$

Proof:

(a) Let A be m x n, B n x p, C p x q. To prove for the case $m=2,\ n=3,\ p=4,\ q=3$.

Let $A=[a_{ij}],\ B=[b_{ij}],\ C=[c_{ij}],\ AB=D=[d_{ij}],\ BC=E=[e_{ij}]$,

$(AB)C=F=[f_{ij}]$, and $A(BC)=G=[g_{ij}]$.

To show: $f_{ij}=g_{ij}$ for all i, j.

Now,

$$f_{ij}=\sum_{k=1}^{4}d_{ik}c_{kj}=\sum_{k=1}^{4}\left(\sum_{r=1}^{3}a_{ir}b_{rk}\right)c_{kj}$$

and

$$g_{ij}=\sum_{r=1}^{3}a_{ir}e_{rj}=\sum_{r=1}^{3}a_{ir}\left(\sum_{k=1}^{4}b_{rk}c_{kj}\right).$$

Then,

$$\begin{aligned}f_{ij}&=\sum_{k=1}^{4}(a_{i1}b_{1k}+a_{i2}b_{2k}+a_{i3}b_{3k})c_{kj}\\ &=a_{i1}\sum_{k=1}^{4}b_{1k}c_{kj}+a_{i2}\sum_{k=1}^{4}b_{2k}c_{kj}+a_{i3}\sum_{k=1}^{4}b_{3k}c_{kj}\\ &=\sum_{r=1}^{3}a_{ir}\left(\sum_{k=1}^{4}b_{rk}c_{kj}\right)=g_{ij}\end{aligned}$$

(b) HW (Exercise 4)

(c) HW (Exercise 4)

Example 3

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 2 \end{bmatrix} \quad 2 \times 3$$

$$B = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & -3 & 1 \end{bmatrix} \quad 3 \times 4$$

$$C = \begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix} \quad 4 \times 3$$

$$A(BC) = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 \\ 4 & 1 & 3 \\ 2 & -4 & -6 \end{bmatrix} = \begin{bmatrix} 2 & 5 & 9 \\ 5 & -5 & -7 \end{bmatrix}$$

$$(AB)C = \begin{bmatrix} 1 & -1 & 4 & 1 \\ 1 & 1 & -4 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 5 & 9 \\ 5 & -5 & -7 \end{bmatrix}$$

Theorem <u>Properties of Scalar Multiplication</u>

If r and s are real numbers and A and B are matrices of the appropriate sizes, then

(a) $r(sA) = (rs)A$

(b) $(r+s)A = rA + sA$

(c) $r(A+B) = rA + rB$

(d) $A(rB) = r(AB) = (rA)B$

Proofs: Exercise 13, 14, 16, 18

Example 4

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 3 & 0 \\ 2 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$3(2A) = 3\begin{bmatrix} 2 & 0 & 2 \\ 0 & 4 & -2 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 6 \\ 0 & 12 & -6 \end{bmatrix} = 6A$$

$$A(2B) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & 6 & 0 \\ 4 & -2 & 2 \\ 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 8 & 0 \\ 8 & -6 & 4 \end{bmatrix} = 2(AB)$$

$$AB = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 2 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 0 \\ 4 & -3 & 2 \end{bmatrix}$$

Theorem <u>Properties of the Transpose</u>

If r is a scalar and A and B are matrices, then

(a) $\left(A^T\right)^T = A$

(b) $(A+B)^T = A^T + B^T$

(c) $(AB)^T = B^T A^T$

(d) $(rA)^T = rA^T$

Proof of (c):

Let $A = [a_{ij}]$ and $B = [b_{ij}]$; $AB = [c_{ij}]$

To prove c_{ij}^T is the (i, j) entry in $B^T A^T$.

$$c_{ij}^T = c_{ji} = \sum_{k=1}^{n} a_{jk} b_{ki} = \sum_{k=1}^{n} a_{kj}^T b_{ik}^T = \sum_{k=1}^{n} b_{ik}^T a_{kj}^T = (i, j) \text{ entry in } B^T A^T .$$

Proof of (a), (b), (d) : Exercises 26 and 27

Example 5

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 3 \\ 0 & -1 \\ 2 & 1 \end{bmatrix}, \quad B^T = \begin{bmatrix} 2 & 0 \\ -1 & 1 \\ 1 & -2 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 3 & -1 & 3 \\ 3 & 0 & -1 \end{bmatrix}$$

$$(A + B)^T = \begin{bmatrix} 3 & 3 \\ -1 & 0 \\ 3 & -1 \end{bmatrix} = A^T + B^T$$

Example 6

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 1 & -1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 3 & -2 \\ 4 & -3 \end{bmatrix}, \quad (AB)^T = \begin{bmatrix} 3 & 4 \\ -2 & -3 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 3 \\ 0 & -1 \\ 2 & 1 \end{bmatrix}, \quad B^T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} 3 & 4 \\ -2 & -3 \end{bmatrix} = (AB)^T$$

Note1. For real numbers *a* and *b*,

$$ab = 0 \Rightarrow a = 0 \text{ or } b = 0 .$$

This is not true for matrices.

Example 7

$$A = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}, \;\; B = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$$

$$A \neq O, \;\; B \neq O, \text{ but } AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Note2. If a, b, c are real numbers for which $ab = ac$ and $a \neq 0$, then $b = c$.
This is <u>not</u> true for matrices.

Example 8

$$A = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}, \;\; B = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}, \;\; C = \begin{bmatrix} 2 & 4 \\ 1 & 5 \end{bmatrix}$$

$$AB = \begin{bmatrix} 3 & 3 \\ -6 & -6 \end{bmatrix} = AC, \;\; \text{but } B \neq C.$$

Summary of the differences between matrix multiplication and multiplication of real numbers:

1. AB need not equal BA.
2. AB may be the zero matrix with $A \neq O$ and $B \neq O$.
3. AB may equal AC with $B \neq C$.