Math 270 Linear Algebra

Chapter 1 Linear Equations and Matrices

1.3 Matrix Multiplication

<u>**Definition**</u> The <u>**dot product**</u> or <u>**inner product**</u> of the *n*-vectors in $\mathbb{R}^n$

$$\vec{a}=\begin{bmatrix}a_1\\a_2\\ \\a_n\end{bmatrix} \text{ and } \vec{b}=\begin{bmatrix}b_1\\b_2\\ \\b_n\end{bmatrix}$$

is defined as

$$\vec{a}\bullet\vec{b}=a_1b_1+a_2b_2+\ldots+a_nb_n=\sum_{i=1}^{n}a_ib_i$$

Example 1 $\vec{u}=\begin{bmatrix}1\\-2\\3\end{bmatrix}$, $\vec{v}=\begin{bmatrix}0\\4\\1\end{bmatrix}\Rightarrow\vec{u}\bullet\vec{v}=(1)(6)+(-2)(4)+(3)(1)=0-8+3=-5$

Example 2 Exercise #4 p. 30 Determine the value of *x* so that $\vec{v}\bullet\vec{w}=0$, where

$$\vec{v}=\begin{bmatrix}1\\-3\\4\\x\end{bmatrix} \text{ and } \vec{w}=\begin{bmatrix}x\\2\\-1\\1\end{bmatrix}.$$

Solution:

$$\vec{v}\bullet\vec{w}=(1)(x)+(-3)(2)+(4)(-1)+(x)(1)=x-6-4+x=2x-10$$
$$\vec{v}\bullet\vec{w}=0\Rightarrow 2x-10=0\Rightarrow 2x=10\Rightarrow x=5$$

Definition If $A=\left[a_{ij}\right]$ is an *m* x *p* matrix and $B=\left[b_{ij}\right]$ is a *p* x *n* matrix, then the product of *A* and *B*, denoted by *AB*, is the *m* x *n* matrix $C=\left[c_{ij}\right]$, defined by

$$c_{ij}=a_{i1}b_{1j}+a_{i2}b_{2j}+\ldots+a_{ip}b_{pj}=\sum_{k=1}^{p}a_{ik}b_{kj}\quad(1\le i\le m,\ 1\le j\le n)\qquad(1)$$

Note:

1. The i,j^{th} element in the product matrix is the dot product of the transpose of the i^{th} row, $row_i(A)$, of *A* and the j^{th} column, $col_j(B)$, of *B*.

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1p} \\ a_{21} & a_{22} & \dots & a_{2p} \\ & & & \\ a_{i1} & a_{i2} & \dots & a_{ip} \\ & & & \\ a_{m1} & a_{m2} & \dots & a_{mp} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1j} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2j} & \dots & b_{2n} \\ & & \dots & & \dots & \\ b_{p1} & b_{p2} & \dots & b_{pj} & \dots & b_{pn} \end{bmatrix}$$

$$= \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ & & c_{ij} & \\ c_{m1} & c_{m2} & \dots & c_{mn} \end{bmatrix}$$

$$c_{ij} = \left(row_i\left(A\right)\right)^T \bullet col_j\left(B\right)$$

2. The product of *A* and *B* is defined only when the number of rows of *B* is exactly the same as the number of columns of *A*.

$${}_mA_p \ {}_pB_n = {}_mAB_n$$

Example 3

$$A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \\ 2 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \end{bmatrix}$$

$$AB = \begin{bmatrix} (1)(1)+(2)(0) & (1)(0)+(2)(1) & (1)(-2)+(2)(-1) \\ (3)(1)+(-1)(0) & (3)(0)+(-1)(1) & (3)(-2)+(-1)(-1) \\ (2)(1)+(-2)(0) & (2)(0)+(-2)(1) & (2)(-2)+(-2)(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & -4 \\ 3 & -1 & -5 \\ 2 & -2 & -2 \end{bmatrix}$$

$$BA = \begin{bmatrix} (1)(1)+(0)(3)+(-2)(2) & (1)(2)+(0)(-1)+(-2)(-2) \\ (0)(1)+(1)(3)+(-1)(2) & (0)(2)+(1)(-1)+(-1)(-2) \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 6 \\ 1 & 1 \end{bmatrix}$$

Note: *AB* and *BA* are both defined but $AB \neq BA$.

Example 4 If *A* is 2 x 3 and *B* is 3 x 5, then *AB* is 2 x 5 and *BA* is defined.

Example 5 Exercise #10 p. 31

Let $A = \begin{bmatrix} 1 & 2 & x \\ 3 & -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} y \\ x \\ 1 \end{bmatrix}$. If $AB = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$, find *x* and *y*.

Solution:

$$AB = \begin{bmatrix} y+2x+x \\ 3y-x+2 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \end{bmatrix} \Leftrightarrow \begin{array}{r} y+3x=6 \\ 3y-x=6 \end{array} \Leftrightarrow \begin{array}{r} -3y-9x=-18 \\ \underline{\;3y-x=6\;} \\ -10x=-12 \\ x=\dfrac{12}{10}=\dfrac{6}{5} \end{array}$$

$$y+3\cdot\frac{6}{5}=6 \Leftrightarrow 5y+18=30 \Leftrightarrow 5y=12 \Leftrightarrow y=\frac{12}{5}$$

Application

Exercise # 50 p. 33

Medicine: diet research project; adults (M and F), children (M and F)

$$A = \begin{array}{c} \begin{array}{cc} \text{adults} & \text{children} \end{array} \\ \begin{bmatrix} 80 & 120 \\ 100 & 200 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{male} \\ \text{female} \end{array} \qquad B = \begin{array}{c} \begin{array}{ccc} \text{protein} & \text{fat} & \text{carb} \end{array} \\ \begin{bmatrix} 20 & 20 & 20 \\ 10 & 20 & 30 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{adult} \\ \text{child} \end{array}$$

a. How many grams of protein are consumed daily by the males in the project?
b. How many grams of fat are consumed daily by the females in the project?

Solution:
a.

$$AB = \begin{bmatrix} (80)(20)+(120)(10) & (80)(20)+(120)(20) & (80)(20)+(120)(30) \\ (100)(20)+(200)(10) & (100)(20)+(200)(20) & (100)(20)+(200)(30) \end{bmatrix}$$

protein consumed by the males:

$$(80)(20)+(120)(10)=1600+1200=2800 \text{ g}$$

b. fat consumed by the females: $(100)(20)+(200)(20)=2000+4000=6000$ g

To find a column in the product *AB* without carrying out the multiplication, use:

Exercise #46a: The j^{th} column of *AB* is the matrix $A\vec{b}_j$, or $A\ \text{col}_j(B)$.

Example 7

$$A = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 3 & 2 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -1 & 2 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & -1 & 2 \end{bmatrix}$$

The 3^{rd} column of *AB* is

$$AB_3 = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 3 & 2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 1 \end{bmatrix}.$$

Note: AB_3 can be written as the linear combination of the columns of A, i.e.,

$$AB_3 = 2\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + (-1)\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

In general, let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ be an m x n matrix and let $\vec{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$

be an n-vector. Then

$$\begin{aligned} A\vec{c} &= c_1\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + c_2\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \ldots + c_n\begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \\ &= c_1A_1 + c_2A_2 + \ldots + c_nA_n \end{aligned}$$

Remark If $\vec{u}$ and $\vec{v}$ are n-vectors, then $\vec{u}\bullet\vec{v} = \vec{u}^T\vec{v}$.
Proof: Exercise 41

Back to linear systems:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad\quad \vdots \quad &\;\; \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \qquad (2)$$

Define

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

Then the linear system can be written in matrix form as

$$A\vec{x} = \vec{b},$$

where A is called the **<u>coefficient matrix</u>** of (2) and the matrix

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \end{array}\right]$$

is called the **augmented matrix** of (2).

Example 8

$$\begin{aligned} x_1 - 3x_2 + x_3 + 4x_4 &= 10 \\ 2x_1 \qquad + 2x_3 - x_4 &= 7 \\ 5x_1 + x_2 \qquad + 3x_4 &= 2 \end{aligned}$$

In matrix form:

$$\begin{bmatrix} 1 & -3 & 1 & 4 \\ 2 & 0 & 2 & -1 \\ 5 & 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 10 \\ 7 \\ 2 \end{bmatrix}$$

The coefficient matrix is

$$\begin{bmatrix} 1 & -3 & 1 & 4 \\ 2 & 0 & 2 & -1 \\ 5 & 1 & 0 & 3 \end{bmatrix}$$

and the augmented matrix is

$$\left[\begin{array}{cccc:c} 1 & -3 & 1 & 4 & 10 \\ 2 & 0 & 2 & -1 & 7 \\ 5 & 1 & 0 & 3 & 2 \end{array}\right].$$

Example 9 Given the augmented matrix

$$\left[\begin{array}{cccc:c} 1 & 0 & 3 & -2 & 5 \\ 0 & 2 & -1 & 4 & 6 \end{array}\right],$$

then the corresponding linear system is

$$\begin{aligned} x_1 \qquad + 3x_3 - 2x_4 &= 5 \\ 2x_2 - x_3 + 4x_4 &= 6 \end{aligned}$$

Note: $A\vec{x}$ can be written as a linear combination of the columns of A:

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \\ a_{m2} \end{bmatrix} + \ldots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \\ b_m \end{bmatrix} \qquad (3)$$

(3) provides another way of thinking about the solution of a linear system:

$A\vec{x} = \vec{b}$ is consistent if and only if $\vec{b}$ can be expressed as a linear combination of the columns of the matrix A.