Math 270 Linear Algebra

Chapter 1 Linear Equations and Matrices

1.2 Matrices

<u>Definitions</u>

An $m \times n$ matrix A is a rectangular array of mn numbers arranged in m horizontal rows and n vertical columns.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \qquad (1)$$

The ith row of A is

$$\begin{bmatrix} a_{i1} & a_{i2} & \dots & a_{in} \end{bmatrix}, \quad 1 \le i \le m,$$

and the jth column of A is

$$\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}, \; 1 \le j \le n.$$

If $m = n$, A is called a **<u>square matrix of order *n*</u>** and $a_{11}, a_{22}, \dots, a_{nn}$ form the **<u>main diagonal</u>** of *A*. a_{ij} is called the (i, j) **<u>entry</u>** of *A* or **<u>i,jth element</u>** of *A* and we write

$$A = \left[a_{ij} \right].$$

Example 1

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \;\; 2 \times 3; \quad B = \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix} \;\; 3 \times 1; \quad C = \begin{bmatrix} -1 & 1 & 0 & 2 \end{bmatrix} \;\; 1 \times 4;$$

$$D = \begin{bmatrix} 1 & 0 & 2 \\ 5 & 7 & 8 \\ 0 & -5 & 2 \end{bmatrix} \;\; 3 \times 3; \; \text{1, 7,2 are on the main diagonal}$$

Definition An n x 1 matrix is also called an **<u>*n*-vector</u>**.

Ex. $B = \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}$ is a **<u>3-vector</u>** and we write $\vec{u} = \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}$.

Notation: $\mathbb{R}^n$ set of n-vectors with real entries
$\mathbb{C}^n$ set of n-vectors with complex entries

Definition Two m x n matrices $A = \left[a_{ij}\right]$ and $B = \left[b_{ij}\right]$ are **equal** if they agree entry by entry, i.e., if $a_{ij} = b_{ij}$ for $i = 1,\ldots,m$ and $j = 1,\ldots,n$.

Example 2

$$A = \begin{bmatrix} 2 & 0 \\ 4 & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} x-2 & y \\ 4 & z \end{bmatrix} \text{ are equal iff } x-2=2 \text{ or } x=4,\ y=0, \text{ and } z=-1.$$

Matrix Operations

Definition If $A = \left[a_{ij}\right]$ and $B = \left[b_{ij}\right]$ are both m x n matrices, then the **sum** $A + B$ is an m x n matrix $C = \left[c_{ij}\right]$ defined by $c_{ij} = a_{ij} + b_{ij}$, $i = 1,\ldots,m$ and $j = 1,\ldots,n$.

Example 3

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \end{bmatrix},\ B = \begin{bmatrix} 0 & -1 & 5 \\ 1 & 3 & 4 \end{bmatrix} \Rightarrow A + B = \begin{bmatrix} 1 & 1 & 8 \\ 0 & 1 & 1 \end{bmatrix}$$

Definition If $A = \left[a_{ij}\right]$ is an m x n matrix and r is a real number, then the **scalar multiple** of A by r, rA, is the m x n matrix $C = \left[c_{ij}\right]$, where $c_{ij} = ra_{ij}, i = 1,\ldots,m; j = 1,\ldots,n$

Example 4

$$3\begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 9 \\ -3 & -6 & -9 \end{bmatrix}$$

Definition The difference between A and B is $A + (-1)B$ and is written as $A - B$.

Example 5

$$A = \begin{bmatrix} 3 & -2 \\ 5 & 7 \\ 0 & 4 \end{bmatrix},\ B = \begin{bmatrix} -3 & 5 \\ 9 & -2 \\ -1 & 1 \end{bmatrix} \Rightarrow A - B = \begin{bmatrix} 6 & -7 \\ -4 & 9 \\ 1 & 3 \end{bmatrix}$$

Application #20 p. 20

large steel manufacturer has 2000 employees
each employee's salary is a component of a vector $\vec{u}$ in $\mathbb{R}^{2000}$
8% across the board salary increase
Express new salaries using $\vec{u}$.

Solution:

New salaries = $\vec{u} + 0.08\vec{u} = 1.08\vec{u}$

Summation Notation

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \ldots + a_n$$

i **index of summation**; dummy variable

$$\sum_{i=1}^{n} a_i = \sum_{j=1}^{n} a_j = \sum_{k=1}^{n} a_k$$

Properties

1. $\sum_{i=1}^{n} (r_i + s_i) a_i = \sum_{i=1}^{n} r_i a_i + \sum_{i=1}^{n} s_i a_i$

2. $\sum_{i=1}^{n} c(r_i a_i) = c \sum_{i=1}^{n} r_i a_i$

3. $$\underbrace{\sum_{j=1}^{n} \underbrace{\left(\sum_{i=1}^{m} a_{ij} \right)}_{\text{add all entries in each column}}}_{\text{add all the numbers}} = \underbrace{\sum_{i=1}^{m} \underbrace{\left(\sum_{j=1}^{n} a_{ij} \right)}_{\text{add all entries in each row}}}_{\text{add all the numbers}}$$

Definition If $A_1, A_2, \ldots, A_k$ are m x n matrices and $c_1, c_2, \ldots, c_k$ are real numbers, then an expression of the form

$$c_1 A_1 + c_2 A_2 + \ldots + c_k A_k \qquad (2)$$

is called a **linear combination** of $A_1, A_2, \ldots, A_k$ and $c_1, c_2, \ldots, c_k$ are called **coefficients**.
(2) can also be written as

$$\sum_{i=1}^{k} c_i A_i = c_1 A_1 + c_2 A_2 + \ldots + c_k A_k$$

Example 6

$$2\begin{bmatrix} 0 & -1 & 1 \\ 3 & 5 & -2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -2 & -1 \\ 9 & 7 & -7 \end{bmatrix}$$

Definition If $A = \left[a_{ij} \right]$ is an m x n matrix, then the **transpose** of A, $A^T = \left[a_{ij}^T \right]$, is the n x m matrix defined by $a_{ij}^T = a_{ji}$.

Example 7 $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$