Math 270 Linear Algebra

Chapter 1 Linear Equations and Matrices

1.1 Systems of Linear Equations

<u>Definitions</u>

The equation

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \qquad (1)$$

where $a_1, a_2, \ldots, a_n$, and b are real or complex constants and $x_1, x_2, \ldots, x_n$ are unknowns, is called a **<u>linear equation</u>**. A **<u>solution</u>** to (1) is a sequence of numbers $s_1, s_2, \ldots, s_n$ such that (1) is satisfied when $x_1 = s_1, x_2 = s_2, \ldots, x_n = s_n$ are substituted in (1).

A **<u>system of *m* linear equations in *n* unknowns</u>** $x_1, x_2, \ldots, x_n$ or a **<u>linear system</u>**, is a set of *m* linear equations in *n* unknowns, and can be written as

$$\begin{array}{llll} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array} \qquad (2)$$

The *i*th equation is

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i .$$

In (2), the a_{ij} are known constants and given values of $b_1, b_2, \ldots, b_n$, we want to find the values of $x_1, x_2, \ldots, x_n$ that will satisfy each equation in (2).

A **<u>solution</u>** to a linear system (2) is a sequence of *n* numbers $s_1, s_2, \ldots, s_n$, which has the property that each equation in (2) is satisfied when $x_1 = s_1, x_2 = s_2, \ldots, x_n = s_n$ are substituted.

If (2) has no solution, it is called an **<u>inconsistent</u>** system; otherwise, it is called **<u>consistent</u>**. If $b_1 = b_2 = \ldots = b_m = 0$, then (2) is called a **<u>homogeneous system</u>**. Note that $x_1 = x_2 = \ldots = x_n = 0$ is always a solution and is called the **<u>trivial solution</u>**. A solution in which not all of $x_1, x_2, \ldots, x_n$ are zero is called a **<u>nontrivial solution</u>**.

Consider another system of *r* linear equations in *n* unknowns:

$$\begin{array}{llll} c_{11}x_1 + c_{12}x_2 + \cdots + c_{1n}x_n = d_1 \\ c_{21}x_1 + c_{22}x_2 + \cdots + c_{2n}x_n = d_2 \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ c_{r1}x_1 + c_{r2}x_2 + \cdots + c_{rn}x_n = d_r \end{array} \qquad (3)$$

(2) and (3) are said to be **<u>equivalent</u>** if they both have exactly the same solution.

Example 1

$$\begin{aligned} 2x_1 + 3x_2 &= 4 \\ -x_1 - x_2 &= -1 \end{aligned} \tag{4}$$

is equivalent to

$$\begin{aligned} 3x_1 - x_2 &= -5 \\ x_1 + x_2 &= 1 \\ 5x_1 + 7x_2 &= 9 \end{aligned} \tag{5}$$

and both have $x_1 = -1$ and $x_2 = 2$ for solutions.

Method of Elimination

3 cases:	$m = n$	"square"
	$m > n$	"tall"
	$m < n$	"long"

Example 2

$$\begin{aligned} x_1 + x_2 &= 3 \\ 3x_1 - 2x_2 &= -1 \end{aligned} \tag{6}$$

Eliminate x_2: 2(1st) + (2nd)

$$\begin{aligned} 2x_1 + 2x_2 &= 6 \\ 3x_1 - 2x_2 &= -1 \\ \hline 5x_1 \qquad &= 5 \Rightarrow x_1 = 1 \end{aligned}$$

subsitute in the 1st:

$$1 + x_2 = 3 \Rightarrow x_2 = 2$$

consistent system

Example 3

$$\begin{aligned} x + y &= 3 \\ 3x + 3y &= 5 \end{aligned} \tag{7}$$

Eliminate y: −3 (1st) + (2nd):

$$\begin{aligned} -3x - 3y &= -9 \\ 3x + 3y &= 5 \\ \hline 0 &= -4 \end{aligned}$$

inconsistent system

Example 4 #2 p.8

$$\begin{aligned} 2x-3y+4z &= -12 \\ x-2y+z &= -5 \\ 3x+y+2z &= 1 \end{aligned} \qquad (8)$$

Eliminate *x*: $1^{st} + (-2)\,2^{nd}$:

$$\begin{array}{rcr} 2x-3y+4z &=& -12 \\ -2x+4y-2z &=& 10 \\ \hline y+2z &=& -2 \end{array}$$

Eliminate *x* ***again***: $(-3)\,2^{nd} + 3^{rd}$:

$$\begin{array}{rcr} -3x+6y-3z &=& -12 \\ 3x+y+2z &=& 1 \\ \hline 7y-z &=& 16 \end{array}$$

We get the reduced system:

$$\begin{aligned} y+2z &= -2 \\ 7y-z &= 16 \end{aligned} \qquad (9)$$

Eliminate *y*: $(-7)\,1^{st} + 2^{nd}$:

$$\begin{array}{rcr} -7y-14z &=& 14 \\ 7y-z &=& 16 \\ \hline -15z &=& 30 \end{array}$$

$-15z = 30 \Rightarrow z = -2$ substitute in 1st of (9)

$$y + 2(-2) = -2 \Rightarrow y = 2$$

Substitute *z* and *y* in 1st of (8):

$$2x - 3(2) + 4(-2) = -12 \Rightarrow 2x - 6 - 8 = -12 \Rightarrow x = 1$$

Note: The elimination procedure produced the system

$$\begin{aligned} 2x-3y+4z &= -12 \\ y+2z &= -2 \\ z &= -2 \end{aligned} \qquad (10)$$

(10) is equivalent to (8).

Example 5 #8 p.9

$$\begin{aligned} 3x+4y-z &= 8 \\ 6x+8y-2z &= 3 \end{aligned} \qquad (11)$$

Eliminate x: $(-2)\,1^{st} + 2^{nd}$:

$$\begin{aligned} -6x-8y+2z &= -16 \\ 6x+8y-2z &= 3 \\ \hline 0 &= -13 \end{aligned}$$

(11) is inconsistent.

Example 6 #6 p. 8

$$\begin{aligned} x+y-2z &= 5 \\ 2x+3y+4z &= 2 \end{aligned} \qquad (12)$$

Eliminate x: $(-2)\,1^{st} + 2^{nd}$:

$$\begin{aligned} -2x-2y+4z &= -10 \\ 2x+3y+4z &= 2 \\ \hline y+8z &= -8 \Rightarrow y=-8z-8 \text{ substitute in the 1st} \end{aligned}$$

$x+(-8z-8)-2z=5$

$x-10z-8=5 \Rightarrow x=10z+13$

Thus, the solution is

$x=10z+13$

$y=-8z-8$

z any real number

and (12) has infinitely many solutions.

Example 7 #12 p.9

$$\begin{aligned} x-5y &= 6 \\ 3x+2y &= 1 \\ 5x+2y &= 1 \end{aligned} \qquad (13)$$

Eliminate y: $-1 \times 2^{nd} + 3^{rd}$:

$$\begin{aligned} -3x-2y &= -1 \\ 5x+2y &= 1 \\ \hline 2x &= 0 \Rightarrow x=0 \text{ substitute in the 1st} \end{aligned}$$

$$-5y=6 \Rightarrow y=-\frac{6}{5}$$

But $x=0,\ y=-\frac{6}{5}$ don't satisfy the rest.

Thus, (13) has no solution.

The method of elimination involves **3 manipulations**:

1. Interchange the ith and jth equations.
2. Multiply an equation by a nonzero constant.
3. Replace the ith equation by c times the jth equation plus the ith equation, $i \neq j$, i.e. replace

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i$$

by

$$(ca_{j1} + a_{i1})x_1 + (ca_{j2} + a_{i2})x_2 + \cdots + (ca_{jn} + a_{in})x_n = cb_j + b_i$$

It is *easy to show* that performing these manipulations on a linear system leads to an equivalent system. (Exercise 24, Example 6, Exercise 25)

Example 6 p. 7

Suppose that the ith equation of (2) is multiplied by $c \neq 0$.

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \qquad \vdots \qquad\qquad \vdots \qquad \vdots \\ ca_{i1}x_1 + ca_{i2}x_2 + \cdots + ca_{in}x_n = cb_i \\ \vdots \qquad \vdots \qquad\qquad \vdots \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array} \qquad (14)$$

If $x_1 = s_1,\ x_2 = s_2, \ldots, x_n = s_n$ is a solution to (2), then it is a solution to (14) except possibly for the ith equation:

$$c(a_{i1}s_1 + a_{i2}s_2 + \cdots + a_{in}s_n) = cb_i$$
$$ca_{i1}s_1 + ca_{i2}s_2 + \cdots + ca_{in}s_n = cb_i$$

Thus, the ith equation is also satisfied and every solution of (2) is also a solution to (14).

Conversely, every solution to (14) also satisfies (2).
Hence, (2) and (14) are equivalent systems.

Application #30 p. 10

oil refinery	low-sulfur fuel high-sulfur fuel	
low-sulfur	5 min blending plant 4 min refining plant	3 hours
high-sulfur	4 min blending plant 2 min refining plant	2 hours

How many tons of each type of fuel should be manufactured so that the plants are fully used?

Solution:

Let x_1 = number of tons of l-s fuel

x_2 = number of tons of h-s fuel

$$\begin{aligned} 5x_1 + 4x_2 &= 180 \\ 4x_1 + 2x_2 &= 120 \end{aligned} \qquad (15)$$

Eliminate x_2: 1st + (-2)2nd:

$$\begin{aligned} 5x_1 + 4x_2 &= 180 \\ \underline{-8x_1 - 4x_2} &= \underline{-240} \\ -3x_1 \qquad &= -60 \Rightarrow x_1 = 20 \text{ tons of l-s fuel} \end{aligned}$$

$$5(20) + 4x_2 = 180$$

$$4x_2 = 80 \Rightarrow x_2 = 20 \text{ tons of h-s fuel}$$